# A GENERALIZATION OF TRIGONOMETRIC CONVEXITY AND ITS RELATION TO POSITIVE HARMONIC FUNCTIONS IN HOMOGENEOUS DOMAINS

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ABSTRACT. We consider functions which are subfunctions with respect to the differential operator

$$L_\rho = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2\rho\frac{\partial}{\partial x} + \rho^2$$

and are doubly periodic in $\mathbb{C}$.    These functions play an important role in describing the asymptotic behavior of entire and subharmonic functions of finite order [6, Ch.3]. In studying their properties we are led to problems concerning the uniqueness of Martin functions and the critical value for the parameter $\rho$ in the homogeneous boundary problem for the operator $L_\rho$ in a domain on the torus.

**Introduction.** The relation between entire functions and potential theory is a long-standing theme in complex analysis.

Consider an entire function $f$ of order $\rho$, $0 < \rho < \infty$, mean type, i.e., letting $M(r, f) = \max_\theta \log |f(re^{i\theta})|$, we have that $0 < \limsup_{r\to\infty} r^{-\rho}M(r) < \infty$. The classical Phragmén-Lindelöf indicator corresponding to $f$ is defined as

$$h(\theta)(= h_f(\theta)) = \limsup_{r\to\infty} \frac{\log |f(re^{i\theta})|}{r^\rho},$$

and its key property (cf.[16]) is that it is $2\pi$-periodic and (in the sense of distributions)

$$(0.1) \qquad h''(\theta) + \rho^2 h(\theta) = \nu(d\theta) \geq 0 \qquad (0 \leq \theta \leq 2\pi),$$

where $\nu$ is a (positive) measure. A $2\pi$-periodic function $h$ which satisfies (0.1) is called $\rho$-*trigonometrically convex* ($\rho$–t.c.). The behavior of solutions to (0.1) reveals many facts about entire functions of finite order ([16]). Note that (0.1) holds if and only if

$$(0.2) \qquad v(z) = r^\rho h(\theta)$$

is subharmonic in the plane. To see the connection between (0.2) and entire functions of order $\rho$, mean type, recall that $f$ is of completely regular growth if for some $0 < \rho < \infty$ the following limit exists

$$(0.3) \qquad D' - \lim_{t\to\infty} \log |f(zt)|t^{-\rho},$$

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in the distributional topology $D'(\mathbb{C} \setminus 0)$. Calling this limit $v(z)$ it will then necessarily have the form (0.2) (see [16],[6, Ch.3]).

This class of functions was introduced (in a modified form) independently by B. Ja. Levin and A. Pfluger, and is a major focus of [16]. A more complicated asymptotic behavior arises from the class of entire functions with *periodic limit set* [6, Ch. 3]. This class is based on a *T-automorphic* subharmonic function $v(z)$; i.e., a subharmonic function for which there are fixed $T > 1$, $\rho > 0$ such that

$$(0.4) \qquad\qquad v(Tz) = T^\rho \, v(z) \quad (z \in \mathbb{C}).$$

Given such a function $v$, we thus consider an entire function $f$ which satisfies a condition analogous to (0.3): for every $1 < \tau \le T$ there exists a sequence $t_j \to \infty$ such that

$$D' - \lim_{t_j \to \infty} \log |f(zt_j)| t_j^{-\rho} = v(z\tau)\tau^{-\rho}$$

and for every sequence $u_{t_j}(z) = \log |f(zt_j)| t_j^{-\rho}$ there exists a $\tau$ as above and a subsequence converging to $v(z\tau)\tau^{-\rho}$. If $v$ happens to satisfy (0.4) for every $T > 0$, as in (0.2), then we recover (0.3), because the family $\{u_t(z) : t \in [1, \infty)\}$ is always compact in $D'(\mathbb{C} \setminus 0)$.

The functional equation (0.4) makes it natural to consider functions defined on open sets $G$ which are invariant under multiplication by $T$, i.e., $TG = G$. We call such $G$ a *T-homogeneous set*, and reserve the notion *T-homogeneous domain* to indicate that $G$ is open and connected. The boundary of a nonempty $T$-homogeneous set $G$ (not $\mathbb{C}$) always includes $0$ and $\infty$, and we always assume that $\partial G$ (and $\partial D$, below) has positive capacity.

Consider the class $\mathcal{P}$ of positive harmonic functions on $T$-homogeneous domains $G$ which are bounded in any bounded subset of $G$ and which vanish quasi-everywhere (i.e., outside a set of zero capacity) on $\partial G$ [3, 10, 14, 18]. For a general $T$-homogeneous domain, the class $\mathcal{P}$ may contain infinitely many non-proportional functions (see an example in §3.) We identify a subclass $\mathcal{F} \subset \mathcal{P}$ consisting of functions of restricted growth at infinity, as in (0.6) below, which turns out to be always non-empty and one-dimensional: it consists of positive multiples of a single function. We show in §5 that $\mathcal{P} = \mathcal{F}$ for a large class of domains.

**Theorem 0.5.** *Let $G \subset \mathbb{C}$ be a $T$–homogeneous domain. Let the family $\mathcal{F} \subset \mathcal{P}$ consist of functions $v \in \mathcal{P}$ such that*

$$(0.6) \qquad\qquad M(r, v) \equiv \max_{\substack{|z|=r \\ z \in G}} v(z) \le Cr^k \quad (r > r_0)$$

*for some $r_0 = r_0(v) < \infty$ and $0 \le k = k(v) < \infty$ (i. e., $v$ has finite order). Choose some $z_0 \in G$, $|z_0| = 1$. Then*

(1) *there exists a unique function $H \in \mathcal{F}$ with $H(z_0) = 1$ and hence $v \in \mathcal{F} \Leftrightarrow v = cH$ for some constant $c > 0$;*

(2) *there exists a unique $\rho(G) > 0$ such that every $v \in \mathcal{F}$ satisfies the functional equation*

$$(0.7) \qquad\qquad v(Tz) = T^{\rho(G)} v(z)$$

Let us note that the equation (0.7) coincides with (0.4) for $\rho = \rho(G)$.

Many properties of $G$ are reflected in $\rho(G)$, and will be discussed in, for example, Theorem 0.17, §4, and §6. In §4.6 we present several interpretations of $\rho(G)$ when $G$ is simply-connected and $T$-invariant.

Now let $v$ satisfy (0.4). Then the function

$$(0.8) \qquad q(z) = v(e^z)e^{-\rho x}$$

is $2\pi$–periodic in $y$ and periodic in $x$ with period $P = \log T$. The function $q$ can be considered as a function on a torus $\mathbb{T}_P^2$, obtained by identifying the opposite sides of the rectangle $R = (0, P) \times (-\pi, \pi)$. The homology group of $\mathbb{T}_P^2$ is nontrivial, with basis the cycles $\gamma_1, \gamma_1'$, where $\gamma_1 = \mathbb{T}_P^2 \cap \{y = 0\}$, $\gamma_1' = \mathbb{T}_P^2 \cap \{x = 0\}$.

Let $\pi$ be the covering map of $\mathbb{C}$ onto $\mathbb{T}_P^2$, then $\phi = \pi \circ \log$ is a well-defined covering map of $\mathbb{C} \setminus \{0\}$ onto $\mathbb{T}_P^2$, where the group of deck transformations is given by the dilations by $T^m$ for $m \in \mathbb{Z}$. So if $G$ is a given $T$-homogeneous domain, then

$$(0.9) \qquad D = \pi \circ \log G = \phi(G),$$

is a domain in $\mathbb{T}_P^2$. On the other hand, not every domain in $\mathbb{T}_P^2$ has a $T$-homogeneous domain as its preimage under $\phi$. The preimage $\phi^{-1}(D)$ under $\phi$ is a possibly disconnected set which is invariant under dilations by $T^m$ for $m \in \mathbb{Z}$. An intrinsic description is given by the next proposition.

**Proposition 0.10.** *Let $\hat{\gamma}$ be a closed curve in a domain $D \subset \mathbb{T}_P^2$ homologous in $\mathbb{T}_P^2$ to a cycle $\gamma = n_1\gamma_1 + n_1'\gamma_1'$, $n_1, n_1' \in \mathbb{Z}$. Then*
  *1. If no such $\hat{\gamma}$ can be found in $D$ so that $n_1 \neq 0$, then*

$$\phi^{-1}(D) = \cup_{j=-\infty}^{\infty} G_j,$$

*where $G_j = T^j G_0$, $G_0$ is an arbitrary connected component of $\phi^{-1}(D)$, and $G_j \cap G_l = \emptyset$ for $j \neq l$.*
  *2. If there exists a curve $\hat{\gamma}$ as above with $n_1 \neq 0$, then*

$$\phi^{-1}(D) = \cup_{q=0}^{k-1} G_q,$$

*where $k = \min |n_1|$ with the minimum taken over all such curves $\hat{\gamma}$; $G_0$ is an arbitrary component of $\phi^{-1}(D)$; $G_j$, $j = 0, 1, ..., k-1$, are disjoint $T^k$-homogeneous domains, and for every $m \in \mathbb{Z}$, $T^m G_0 = G_q$, provided $m = lk + q$, for some $q \in \mathbb{Z}$, $0 \leq q \leq k-1$, $l \in \mathbb{Z}$.*

We call domains as in part 2 of Proposition 0.10 *connected on spirals*. In particular, this proposition shows that for every $D$ connected on spirals, we can find a connected $T^k$- homogeneous domain that relates to $D$ by (0.9). The proof of Proposition 0.10 is given in §2.

Let us give examples. The domain $D' = \mathbb{T}_P^2 \cap \{|x - P/2| < P/4\}$ is not connected on spirals, while $D'' = \mathbb{T}_P^2 \cap \{|y| < \pi/4\}$ is. It follows that $D' \cap D''$ is not connected on spirals while $D' \cup D''$ is.

The situation can be more complicated. Let $R$ be the rectangle $[0, P] \times [-\pi, \pi]$. We construct a network of disjoint strips $\{D_j\}_{-\infty}^{+\infty}$ which connect the vertical portions of $\partial R$. If $I_j = D_j \cap \{x = 0\}$ and $I_j' = D_j \cap \{x = P\}$, we arrange that $I_j$

and $I'_{j-1}$ have the same projection on the $y$-axis, and these projections cluster to $\{y = \pm\pi\}$ as $j \to \pm\infty$. After the usual identification of sides of $\partial R$, we obtain a domain $D \subset \mathbb{T}^2_P$ which winds infinitely often in the $x$-direction on $\mathbb{T}^2_P$. This $D$ is not connected on spirals and illustrates how the "bad" case in the proof of Proposition 2.1 looks like. More complicated domains yet are obtained by replacing the fundamental rectangle $R$ by a fundamental parallelogram $R'$ whose vertical sides are $\{x = 0, -\pi < y < \pi\}$ and $\{x = P, -\pi + k2\pi < y < \pi + k2\pi\}$ for some fixed integer $k$. Then, the strips $D'_j$ are in $R'$ and connect $I_j$ to $I''_j$, but now the projections of $I_j$ and $I''_{j-1}$ on the $y$-axis differ by a translation of $k2\pi$ units.

One more example. Consider the family of lines $L_l := \{z = x + iy : y = \pi/(kP)x + l\pi/k, \ x \in \mathbb{R}\}, \ l \in \mathbb{Z}$. It determines a closed curve (spiral) $\hat{\gamma}$ on $\mathbb{T}^2_P$ with $n_1 = k$. The open set $D_k = \{z : |z - \zeta| < \epsilon, \zeta \in L_l, \ l \in \mathbb{Z}\}, \ 0 < \epsilon < P/2\sqrt{\pi^2 + k^2}$, determines a domain $\hat{D}_k$ on $\mathbb{T}^2_P$ that is connected on spirals, and such that $\phi^{-1}(\hat{D}_k)$ consists of $k$ components, each $T^k$-homogeneous.

**0.11.** Since the function $v$ of (0.4) is subharmonic, the function $q$ of (0.8) is upper semicontinuous and in the $D'$ topology on $\mathbb{T}^2_P$ satisfies the inequality $L_\rho q \geq 0$, where

$$(0.12) \qquad\qquad L_\rho := \Delta + 2\rho\frac{\partial}{\partial x} + \rho^2.$$

That is, $L_\rho q$ is a positive measure on $\mathbb{T}^2_P$.

The operator $L_\rho$ arises naturally since (0.8) shows that if $v(Z)$ is a smooth function and $q$ is related to $v$ as in (0.8), then

$$(0.13) \qquad\qquad \Delta_Z v(Z) = e^{(\rho-2)x} L_\rho q(z), \quad Z = e^z, \ z = x + iy.$$

Such functions $q$ are called *subfunctions with respect to* $L_\rho$, or $L_\rho$-*subfunctions*. Note that $L_\rho$ is not symmetric when $\rho \neq 0$.

In this paper we obtain some properties of $L_\rho$-subfunctions; these generalize those of $\rho$-t.c.functions. For the theory of entire functions modeled on functions $v(z)$ as in (0.4), the $L_\rho$-subfunctions play the same role that the $\rho$–t.c.functions play for entire functions of completely regular growth (see [2, 3, 4]).

**0.14.** The study of $L_\rho$-subfunctions depends on properties of the operator $L_\rho$ for arbitrary $\rho$ and, in particular, on properties of solutions to the homogeneous boundary problem

$$(0.15) \qquad\qquad \begin{aligned} L_\rho q &= 0 \quad \text{in } D; \\ q\,\big|_{\partial D} &= 0, \end{aligned}$$

where $D$ is a domain in $\mathbb{T}^2_P$ and $q$ is bounded in $\partial D$ with boundary value zero quasi-everywhere. This is a spectral problem for a *pencil* of differential operators (the standard reference is [17]; cf. §1 below).

We emphasize that in principle a solution of this problem can be defined for an arbitrary domain $D \subset \mathbb{T}^2_P$; recall, however, that the boundaries of all domains considered here have positive capacity.

The *spectrum* of the problem (0.15) consists of those (complex) $\rho$ for which (0.15) holds for some function $q \not\equiv 0$. We identify when the spectrum is nonempty, and give some basic properties in Propositions 1.36, 1.37. We also show that the *minimum positive point* of this spectrum, $\rho(D)$, exists, and is intimately connected with the function $H$ and the number $\rho(G)$ produced in Theorem 0.5, with some component $G$ of $\phi^{-1}(D)$ (see (0.9)).

**Theorem 0.16.** *The following hold:*

(1) $\rho(D) < \infty$ *iff $D$ is connected on spirals;*
(2) *If $\rho(D) < \infty$, then $\rho(D) = \rho(G)$, and up to a constant multiple the corresponding eigenfunction is*

$$q(z) = H(e^z)e^{-\rho(D)x}.$$

A property of $\rho(D)$ which carries over from the classical potential theory is *strict monotonicity*:

**Theorem 0.17.** *Let $D_1$, $D_2$ be domains on $\mathbb{T}_P^2$ which are connected on spirals. If $D_1 \subset D_2$ and $\mathrm{Cap}\,(D_2 \setminus D_1) > 0$ then the strict inequality $\rho(D_1) > \rho(D_2)$ holds.*

**0.18.** In §1 we find the fundamental solution (for $\rho \notin \mathbb{Z}$) and the generalized fundamental solution (for $\rho \in \mathbb{Z}$) of the equation $L_\rho q(z) = 0$ on the whole torus $\mathbb{T}_P^2$. In §8.14 we use it in a representation which is a generalization of the well known representation of $\rho$–t.c. functions from [16, Theorem 24] (see also [6, §2,(4),(5)]). For application of this representation, see [2].

In §7 we introduce the Green function for $L_\rho$, and use this as basis for studying $L_\rho$-subfunctions on subdomains of the torus $\mathbb{T}_P^2$.

We also consider subharmonic minorants of a given real function $m$ in the plane (see, for example, [14]). In application to subharmonic functions with periodic limit sets this leads us to considering of $L_\rho - subminorants\ of\ a\ function\ m$, i. e., $L_\rho-$ subfunctions $u(z)$ with $u(z) \leq m(z)$ for $z \in \mathbb{T}_P^2$. Theorems 9.15 and 9.16 imply

**Theorem 0.19.** *Let $m$ be a continuous function on $\mathbb{T}_P^2$. If $m$ has a non-zero $L_\rho$ subminorant, then $\rho(D) \leq \rho$ for some component $D$ of the set $\mathcal{M}_+ := \{z : m(z) > 0\}$.*

*Conversely, if $\rho(D) < \rho$ (strict inequality!) for some component $D$ of the set $\mathcal{M}_+$, and $m(z) \geq 0$ for all $z \in \mathbb{T}_P^2$, then $m$ has a non-zero $L_\rho$-subminorant.*

This generalizes properties of $\rho$-t.c. functions, since when $D = \{z \in \mathbb{T}_P^2 : \Im z \in (\alpha, \beta)\}$ we have $\rho(D) = \pi/(\beta - \alpha)$.

The borderline case $\rho(D) = \rho$ depends essentially on the behavior of $m$ near $\partial D$, and warrants further scrutiny, as well as the case when $m$ changes its sign.

There is a specific question that arises in studying the completeness of exponential systems [4]. An $L_\rho$–subfunction $u(z)$, $z \in \mathbb{T}_P^2$ is *minimal* if the function $m(z) = u(z) - \epsilon$ does not have an $L_\rho$–subminorant in $\mathbb{T}_P^2$ for arbitrarily small $\epsilon > 0$. A full description of minimal functions is not known (see [8, Problem 16.9]), but some necessary and some sufficient conditions are obtained here. For example, in §9 we show

**Theorem 0.20.** *Let $\mathcal{H}_\rho(u)$ be the maximal open set on which $L_\rho u = 0$. If there exists a connected component $M \subset \mathcal{H}_\rho(u)$ such that $\rho(M) < \rho$, then $u$ is a minimal $L_\rho$-subfunction.*

This paper is organized as follows:

In §1 we study properties of the operator $L_\rho$ and the generalized boundary problem, and prove Theorem 0.5 in §3. Theorem 0.16 and 0.17 are proved in §6, and §§7-9 are devoted to $L_\rho$-subfunctions and subminorants.

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**1. The operator $L_\rho$; characterization of Spec $D$.** First we study fundamental solutions of $L_\rho$ on the whole torus $\mathbb{T}^2_P$.

**Proposition 1.1.** *If $\rho \notin \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, the operator $L_\rho$ has a unique fundamental solution $E_\rho(z)$ on $\mathbb{T}^2_P$ with singularity at $0$ ($+kP + 2\pi l i,\ k, l \in \mathbb{Z}$).*

*Proof.* We solve the equation $L_\rho E_\rho = \delta_0(z)$ in $\mathcal{D}'(\mathbb{T}^2_P)$ (the space of distributions), where $\delta_0$ is the Dirac function supported at $0 \in \mathbb{T}^2_P$. Using the transformation $x' = 2\pi x/P$, $y' = y$, we obtain that the period in $x'$ is $2\pi$, so that $\mathbb{T}^2_P \equiv \mathbb{T}^2_{2\pi} \equiv \mathbb{T}^2$, and $L_\rho$ is replaced by

$$(1.2) \qquad L_{\rho,P} = \left(\frac{2\pi}{P}\right)^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2\rho\left(\frac{2\pi}{P}\right)\frac{\partial}{\partial x} + \rho^2.$$

The Dirac function is characterized by the action $\langle \delta_0, g \rangle = g(0)$ for $g \in \mathcal{D}(\mathbb{T}^2)$, the class of infinitely-differentiable doubly $2\pi$-periodic functions. The system $\phi_{k\ell}(z) = e^{ikx}e^{i\ell y}$ ($k, \ell \in \mathbb{Z}$) is dense in $\mathcal{D}(\mathbb{T}^2)$. We compute the Fourier coefficients of the solution $E$, corresponding to (1.2) by solving

$$(1.3) \qquad \langle L_\rho E, \phi_{k\ell} \rangle \equiv \langle E, L^*_\rho \phi_{k,\ell} \rangle = 1, \qquad (k, \ell \in \mathbb{Z}),$$

where $L^*_{\rho,P} = L_{-\rho,P}$ is symmetric to $L_{\rho,P}$. Equations (1.2) and (1.3) then yield that

$$(1.4) \qquad a_{k\ell} \equiv \langle E, \phi_{k\ell} \rangle = \left[-\left(\frac{2\pi}{P}\right)^2 k^2 - \ell^2 + 2\rho\frac{2\pi}{P}ik + \rho^2\right]^{-1},$$

and since $\rho \notin \mathbb{Z}$, these coefficients are uniquely determined.

Note that if $E_1$ and $E_2$ are solutions, then all Fourier coefficients of $E_1 - E_2$ vanish. Thus $E$ is unique, and

$$E(z) = \sum_{k,\ell} a_{k\ell} e^{ikx} \cdot e^{i\ell y},$$

where $a_{k\ell}$ are determined by (1.4). The series defining $E$ always converges in the sense of distributions, and it is well-known that solutions to elliptic homogeneous equations with constant coefficients are real-analytic (cf. [11]). Thus $E$ is smooth when $z \neq 0$ and has logarithmic singularity at $z = 0$.

Finally, we set

$$E_\rho(z) := E\left(\frac{xP}{2\pi} + iy\right).$$

$\square$

For nonintegral $\rho$ we may express the fundamental solution in another form, using common notions from the theory of subharmonic functions, which also provides an independent way to check the regularity of $E_\rho$ off the diagonal. Let $p = [\rho]$ and $H(u, p)$ be the logarithm of the classical Weierstrass factor of genus $p$ :

$$(1.5) \qquad H(u, p) = \log|1 - u| + \Re\left(\sum_{k=1}^{p} \frac{u^k}{k}\right),$$

and set

$$(1.6) \qquad g(z,p,\rho) = H(e^z,p)\,e^{-\rho x}; \qquad g(z,p,\rho,P) = \sum_{k=-\infty}^{\infty} g(z+kP,p,\rho).$$

This series converges for all $z \neq kP$, $k \in \mathbb{Z}$ because of the inequalities (see, for example, [9, Lemma 1.5])

$$(1.7) \qquad |H(u,p)| \leq \begin{cases} C_{p,\varepsilon}|u|^p, & \text{for } |u| \geq 1+\varepsilon \\ C_{p,\varepsilon}|u|^{p+1}, & \text{for } |u| \leq 1-\varepsilon, \\ C_{p,\varepsilon}, & \text{for } 1-\varepsilon \leq |u| \leq 1+\varepsilon, |u-1| > \varepsilon, \end{cases}$$

where $C_{p,\varepsilon}$ is a constant independent of $u$. Thus $g(z,p,\rho,P)$ is $2\pi$-periodic in $y$ and $P$-periodic in $x$, and so may be viewed as a function on $\mathbb{T}_P^2$.

**Proposition 1.8.** *Let $g(z,p,\rho,P)$ be from (1.6) and $E_\rho$ be the fundamental solution given by Proposition 1.1. Then*

$$\frac{1}{2\pi}\, g\,(z,p,\rho,P) = E_\rho(z).$$

*Proof.* We show that $(1/2\pi)g$ is a fundamental solution of $L_\rho$. To prove this we can use test functions from $\mathcal{D}(\mathbb{T}_P^2)$ supported only near one singularity of the series in (1.6). We can suppose also that it is the term for $k=0$.

Since all other terms satisfy (0.15) , we need only show that

$$(1.9) \qquad\qquad L_\rho g(\cdot,p,\rho) = 2\pi\delta_0$$

on functions $F \in \mathcal{D}((-P/2,P/2)\times(-\pi,\pi))$. Our arguments use the correspondence (0.8).

To avoid confusion, we take $z$ for the variable on $\mathbb{T}_P^2$, and $Z$ for regions in the plane (here $Z = e^z$). Thus let $F$ have compact support near $z=0$ viewed as a point in $\mathbb{T}_P^2$. Then using the calculus of distributions with (0.13) and (1.6) we have

$$I := \, < L_\rho g(\cdot,p,\rho), F > = \, < g(\cdot,p,\rho), L_{-\rho}F > = \int L_{-\rho}F(z)g(z,p,\rho)dxdy =$$

$$= \int [e^{(-\rho-2)x}L_{-\rho}F(z)][e^{\rho x}g(z,p,\rho)]e^{2x}dxdy$$

$$= \int e^{(-\rho-2)x}L_{-\rho}F(z)H(e^z)e^{2x}dxdy = \int \Delta_Z(F(z)e^{-\rho x})H(Z)dXdY.$$

However, $(1/2\pi)\log|z-1|$ is the fundamental solution to the Laplace equation at $z=1$, and so if we set $f(Z) = F(z)e^{-\rho x}$ (recall (0.8)), we find that $I = 2\pi f(1) = 2\pi F(0)$, and this gives (1.9).

By uniqueness of the fundamental function on $\mathbb{T}_P^2$ with singularity at zero, we obtain Proposition 1.8. $\square$

When $\rho$ is an integer, the reasoning which gave Proposition 1.1 gives (proof omitted)

**Proposition 1.10.**   *If $\rho \in \mathbb{Z}$, there exists a generalized fundamental solution $E'_\rho(z)$ satisfying the equation*

(1.11)
$$L_\rho \, E'_\rho(z) = \delta_0(z) - 2\cos\rho y$$

*on $\mathbb{T}^2_P$.*
      *The function $E'_\rho(z)$ is defined uniquely up to an addend of the form*

$$A\cos\rho(y - y_0),$$

*where $A$ and $y_0$ are arbitrary.*

      We next define the spectrum of the problem (0.15) for a domain with arbitrary boundary and prove that it is a discrete set without any finite point of condensation (Proposition 1.36). The natural way for this is to transform the problem (0.15) to an integral equation.
      The domain $D \subset \mathbb{T}^2_P$ is a Riemannian space of hyperbolic type and admits a Green function for the Laplace operator on $\mathbb{T}^2_P$. (see, e.g., [1, Ch.10]).
      We will use the local coordinates $z = x + iy$ which preserves not only the sign of the Laplace operator but also the Laplace operator itself. Thus these coordinates preserve harmonicity and subharmonicity of functions as well as their mass distributions.
      We denote the Green function by $g(z, \zeta, D)$ and extend $g$ to the whole $\mathbb{T}^2_P$ by defining $g(z, \zeta, D) = 0$ when $z$ or $\zeta \in \mathbb{T}^2_P \setminus D$. Denote by $\nabla_z$ the gradient operation in $x, y$; $dz = dxdy$ for $z = x + iy$ and $d\zeta = d\xi d\eta$ for $\zeta = \xi + i\eta$.

**Proposition 1.12.**   *Let $D \subset \mathbb{T}^2_P$ be a domain. Then there exist constants $C = C(p)$ such that*

$$\sup_{z \in \mathbb{T}^2_P} \int_{\mathbb{T}^2_P} |g(z, \zeta, D)|^p \, d\zeta \le C \qquad (0 \le p < \infty);$$

$$\max\left\{ \sup_{z \in \mathbb{T}^2_P} \int_{\mathbb{T}^2_P} |\nabla_\zeta g(z, \zeta, D)|^p d\zeta, \; \sup_{\zeta \in \mathbb{T}^2_P} \int_{\mathbb{T}^2_P} |\nabla_\zeta g(z, \zeta, D)|^p dz \right\} \le C \quad (0 \le p < 2),$$

*Proof.* The function $-g(z, \zeta, D)$ is subharmonic in $\mathbb{T}^2_P \setminus \{\zeta\}$ with masses distributed on $\partial D$ and the total mass 1. In a neigborhood of $\zeta$ it is represented in the form $-g(z, \zeta, D) = v(z, \zeta) - \log|z - \zeta|$ where $v$ is a subharmonic function, with its masses distributed on $\partial D$ and the total mass 1. Note, if the neighborhood does not intersect $\partial D$, then $-g(z, \zeta, D)$ is harmonic. As a matter of fact, the mass distribution of $v$ coincides with the harmonic measure of $D$, which does exists even for every Riemannian space of hyperbolic type.
      It is possible to check, using the Hölder inequality and the continuity in $z$ of the functions

$$\int_U |\log|z - \zeta||^p d\zeta, \; 0 \le p < \infty; \; \int_U |z - \zeta|^p d\zeta, \; 0 \le p < 2,$$

where $U$ is a small disc, that a potential of bounded masses belongs locally to $L^p$, $0 \le p < \infty$, while its gradient belongs locally to $L^p$, $0 \le p < 2$. These are

locally true for every subharmonic function, by the Riesz representation . This also holds on the whole $\mathbb{T}_P^2$ because of its compactness. Thus this is fulfilled for the Green function and its gradients $\nabla_z$ and $\nabla_\zeta$. $\square$

Using $\zeta = \xi + i\eta$ as the local coordinates on $\mathbb{T}_P^2$ we set

$$(1.13) \qquad g^1(z, \zeta, D) := (\partial g / \partial \xi)(z, \zeta, D).$$

The functions $g$ and $g^1$ induce integral operators on $C^\infty(\mathbb{T}_P^2)$, which will be the focus of our attention:

$$G_D q(z) = \int_D g(z, \zeta, D) q(\zeta) d\zeta \equiv \int_{\mathbb{T}_P^2} g(z, \zeta, D) q(\zeta) d\zeta,$$

$$(1.14)$$

$$G_D^1 q(z) = - \int_D g^1(z, \zeta, D) q(\zeta) d\zeta \equiv - \int_{\mathbb{T}_P^2} g^1(z, \zeta, D) q(\zeta) d\zeta.$$

We will use the following properties of these operators:

**Proposition 1.15.** *Let $G_D$, $G_D^1$ be as above. Then $G_D, G_D^1$ can be extended as compact operators from $L^2(\mathbb{T}_P^2)$ to $L^p(\mathbb{T}_P^2)$ for every $p \geq 1$ (in particular, for $p = 2$).*

Before proving this theorem, we present, following [15], some preliminary information on integral operator theory, corresponding to our case.

Set

$$(1.16) \qquad K u(z) := \int_{\mathbb{T}_P^2} K(z, \zeta) u(\zeta) d\zeta.$$

This is an integral operator with the kernel $K(z, \zeta)$, $z, \zeta \in \mathbb{T}_P^2$.

Define the functions

$$(1.17) \qquad \phi_r(z) := \left( \int_{\mathbb{T}_P^2} |K(z, \zeta)|^r d\zeta \right)^{1/r} ;$$

$$(1.18) \qquad \psi_{r^*}(\zeta) := \left( \int_{\mathbb{T}_P^2} |K(z, \zeta)|^{r^*} dz \right)^{1/r^*} ,$$

and their norms

$$(1.19) \qquad \|\phi_r\|_q := \left( \int_{\mathbb{T}_P^2} |\phi_r(z)|^q dz \right)^{1/q} ; \quad \|\psi_{r^*}\|_{q^*} := \left( \int_{\mathbb{T}_P^2} |\psi_{r^*}(\zeta)|^{q^*} d\zeta \right)^{1/q^*} .$$

Set also

$$(1.20) \qquad \|u\|_\alpha := \left( \int_{\mathbb{T}_P^2} |u(z)|^\alpha dz \right)^{1/\alpha} ;$$

and define the space $L^\alpha$ as a space obtained by the closure of the space of infinitely differentiable functions $u(z)$, $z \in \mathbb{T}_P^2$ with respect to this norm. It coincides with the space of mesurable functions for which the integral (1.20) is finite. The space $L^\alpha$ is a Banach space for each $\alpha \geq 1$. We are going to use the following assertion, which is a restatement of [15, Theorem 7.1] for our case.

**Theorem KZPS.** *Suppose $K(z, \zeta)$ satisfies the condition*

$$(1.21) \qquad\qquad \max\{\|\phi_r\|_q, \|\psi_{r^*}\|_{q^*}\} < \infty$$

*for some $r$, $q$, $r^*$, $q^*$ such that*

$$(1.22) \qquad\qquad \min\{r^*, q^*, r\} \geq 1; q \in (0; \infty).$$

*Then for every $0 < \tau < 1$, the integral operator (1.16), acting from $L^{1/\alpha(\tau)}$ to $L^{1/\beta(\tau)}$ for*

$$(1.23) \qquad \alpha(\tau) := 1 - (1 - \tau)(1/r) - \tau(1/q^*); \quad \beta(\tau) := (1 - \tau)(1/q) + \tau(1/r^*),$$

*is compact; and its operator norm satisfies the inequality*

$$(1.24) \qquad\qquad \|K\|_{1/\alpha(\tau) \to 1/\beta(\tau)} \leq \|\phi_r\|_q^{1-\tau} \|\psi_{r^*}\|_{q^*}^{\tau}.$$

**Proposition 1.25.** *Set*

$$(1.26) \qquad\qquad 1/r = 1/r^* = 1/2 + 1/(2p); \quad 1/q = 1/q^* = 1/(2p).$$

*If (1.21) is satisfied under these conditions, then for every $p \geq 1$, the operator $K$ acts from $L^2$ to $L^p$ as a compact operator; and its norm satisfies the inequality (1.24) for $\tau = 1/p$.*

*Proof.* The conditions (1.22) are fulfilled. From (1.23) we have

$$\alpha = 1 - (1 - 1/p)(1/2 + 1/(2p)) - 1/p \ 1/(2p) = 1/2;$$

$$\beta = (1 - 1/p) \ 1/(2p) + 1/p \ (1/2 + 1/(2p)) = 1/p.$$

$\square$

*Proof of Proposition 1.15.* Let us check the conditions (1.21). Set in (1.17), (1.18) $r, q, r^*, q^*$ from (1.26) and $K(z, \zeta) := g(z, \zeta, D)$. We obtain

$$\phi_r(z) = \left( \int_{\mathbb{T}_P^2} |g(z, \zeta, D)|^r d\zeta \right)^{1/r}.$$

Thus $\phi_r(z) \leq C^{1/r}$, $z \in \mathbb{T}_P^2$, by Proposition 1.12 (the first inequality). Using (1.19), we obtain that $\|\phi_r\|_q < \infty$ for every $q$. Since $g(z, \zeta, D) = g(\zeta, z, D)$, $r = r^*$ and $q = q^*$, the inequality $\|\psi_{r^*}\|_{q^*} < \infty$ also holds. Thus the compactness of $G_D$ is proved.

From (1.26) $r, r^* < 2$ if $p \geq 1$. Set $K(z, \zeta) := -g^1(z, \zeta, D)$. In (1.17) we have

$$\phi_r(z) = \left( \int_{\mathbb{T}_P^2} |g^1(z, \zeta, D)|^r d\zeta \right)^{1/r} \leq \left( \int_{\mathbb{T}_P^2} |\nabla_\zeta g(z, \zeta, D)|^r d\zeta \right)^{1/r}.$$

By Proposition 1.12, we obtain $\phi_r(z) \leq C^{1/r}$, $z \in \mathbb{T}_P^2$. Thus $\|\phi_r\|_q < \infty$ holds for every $q$. For proving $\|\psi_{r^*}\|_{q^*} < \infty$, we set

$$\psi_{r^*}(\zeta) = \left( \int\limits_{\mathbb{T}_P^2} |g^1(z, \zeta, D)|^{r^*} dz \right)^{1/r^*}$$

and repeat the previous reasoning. □

Now we may rigorously define problem (0.15) for an arbitrary domain with no assumption concerning its boundary (other than positive capacity).

Consider the *operator pencil*

(1.27) $$\mathcal{G}_D(\rho) := I + 2\rho G_D^1 + \rho^2 G_D;$$

i.e., a family of operators acting from $L^2(\mathbb{T}_P^2)$ to $L^2(\mathbb{T}_P^2)$, depending on a complex parameter $\rho$ (see, e.g., [17]).

Since $q, G_D q, G_D^1 q \in L^2(\mathbb{T}_P^2)$ they also belong to $L^1(\mathbb{T}_P^2)$ and can be considered as distributions in $D$.

**Proposition 1.28.** *Let $D \subset \mathbb{T}_P^2$ be an arbitrary domain (whose boundary has positive capacity), and $q \in L^2(\mathbb{T}_P^2)$. If $q$ satisfies the distribution equation*

(1.29) $$\mathcal{G}_D(\rho)q = 0,$$

*then $q$ is infinitely differentiable inside $D$, $L_\rho q(z) = 0$ for all $z \in D$, $q$ is bounded, and $q$ tends to zero at every regular point of $\partial D$, i.e. $q$ is the solution of (0.15).*

*Proof.* Consider the equality (1.29) in $\mathcal{D}'(D)$. Let $\phi \in \mathcal{D}(D)$ and apply $\mathcal{G}_D q$ to $(1/(2\pi))\Delta\phi$. The definitions of $G_D q$ and $G_D^1 q$ and standard properties of the Laplacian (in particular that $(1/2\pi)\Delta g(\cdot, \zeta) = \delta_\zeta$) imply that $< q, (1/(2\pi))\Delta\phi > = < (1/(2\pi))\Delta q, \phi >$, $< G_D q, (1/(2\pi))\Delta\phi > = < q, \phi >$, as well as

$$< G_D^1 q, \frac{1}{2\pi}\Delta\phi > = -\int_D < \frac{\partial}{\partial\xi}g(\cdot, \zeta), \frac{1}{2\pi}\Delta\phi > q(\zeta)\, d\zeta = -\int_D \frac{\partial}{\partial\xi}\phi(\zeta)q(\zeta)\, d\zeta$$

$$= < \frac{\partial}{\partial\xi}q, \phi > .$$

Hence $0 = < \mathcal{G}_D q, \Delta\phi > = < L_\rho q, \phi >$. By [11, §11] $q$ is infinitely differentiable and satisfies the same equation in the classical sense, so $L_\rho q(z) = 0$ holds in $D$.

We next prove that both integrals which appear in the operator $\mathcal{G}_D q$ vanish as $z \to z_0 \in D$ for any regular point $z_0 \in \partial D$.

It is clear that $z_0$, being regular, ensures that $\lim_{z \to z_0} g(z, \zeta, D) = 0$. Since $g$ is harmonic in $D \setminus \{\zeta\}$, it follows that if $K \subset D$ is compact, we have uniformly that

(1.30) $$\lim_{z \to z_0} g^1(z, \zeta, D) = 0 \quad (\zeta \in K).$$

(recall that $g^1$ means derivative with respect to $\zeta$). We show, for example, that

(1.31) $$\lim_{z \to z_0} G_D^1 q(z) = 0.$$

Proposition 1.15 and the equation $q = -2\rho G^1 q - \rho^2 Gq$ show that $q \in L^p(\mathbb{T}^2_P)$ for some $p > 2$. Let $\epsilon$ be arbitrarily small. Choose $K$ such that

$$(1.32) \qquad \left( \int_{D\backslash K} |q(\zeta)|^p d\zeta \right)^{1/p} < (1/C)\epsilon$$

where $C$ satisfies the condition (Proposition 1.12)

$$(1.33) \qquad \sup_{z\in\mathbb{T}^2_P} \left( \int_{\mathbb{T}^2_P} |\nabla g(z,\zeta)|^{p^*} d\zeta \right)^{1/p^*} \leq C$$

for $p^* = (1 - 1/p)^{-1} < 2$. We have

$$|G^1_D q(z)| \leq \left( \int_{D\backslash K} + \int_K \right) |g^1(z,\zeta,D)q(\zeta)| d\zeta := I_1(z) + I_2(z).$$

Hölder's inequality implies that

$$(1.34) \qquad I_1(z) \leq \left( \int_{D\backslash K} |g^1(z,\zeta,D)|^{p*} d\zeta \right)^{1/p*} \times \left( \int_{D\backslash K} |q(\zeta)|^p d\zeta \right)^{1/p},$$

and so (1.32) and (1.33) yield that $I_1(z) < \epsilon$ for all $z \in D$. For $I_2$ we have that

$$(1.35) \qquad I_2(z) \leq \left( \int_K |g^1(z,\zeta,D)|^{p*} d\zeta \right)^{1/p*} \times \left( \int_K |q(\zeta)|^p d\zeta \right)^{1/p}.$$

Since $q \in L^p$ and (1.30) holds, we obtain that $\lim_{z\to z_0} I_2(z) = 0$, and since $\epsilon$ is arbitrary, these estimates imply (1.31). Let us note that (1.34) and (1.35) imply that $G^1_D q(z)$ is bounded. The equality $\lim_{z\to z_0} G_D q(z) = 0$ and boundedness of $G_D q(z)$ can be obtained in the same way. This and (1.29) show that $\lim_{z\to z_0} q(z) = 0$ and $q$ is bounded. $\square$

Thus (1.29) may be treated as a generalization of the problem (0.15) when $D$ is an *arbitrary domain whose boundary has positive capacity*.

We recall some properties of the operator pencil $\mathcal{G}_D$ (see, e.g. [17, Th.12.9]).

Denote by *Spec D* the *spectrum* of the operator pencil $\mathcal{G}_D$; i.e. the set of $\rho \in \mathbb{C}$ for which the operators $\mathcal{G}_D(\rho)$ have no inverse. In particular, if $\rho \notin Spec\ D$, every solution $q$ to the equation (1.25) is identically zero.

It is essential for applying the cited theorem that the coefficients $G^1_D$ and $G_D$ of the pencil be compact operators and that the spectrum not be the whole plane. The first assertion is Proposition 1.15; while the second is obvious, since the pencil is the identity operator when $\rho = 0$. From the cited theorem we obtain

**Proposition 1.36.** *Let* $\mathcal{G}_D$, $D \subset \mathbb{T}_P^2$, *be the pencil (1.27). Then Spec D is a discrete set (perhaps empty) with no point of accumulation in any finite part of* $\mathbb{C}$.

Here are some special properties of *Spec D*.

**Proposition 1.37.** *Let* $\mathcal{G}_D$, $D \subset \mathbb{T}_P^2$ *be the pencil (1.27) with Spec D* $\neq \emptyset$. *Then*

(1) $\{\rho : \Re\rho = 0\} \cap Spec\ D = \varnothing$;
(2) *The following symmetries hold:*

$$\overline{Spec\ D} := \{\overline{\rho} : \rho \in Spec\ D\} = Spec\ D \pm \frac{2\pi}{P}i = Spec\ D;$$

(3) $Spec(D_-) = -Spec\ D$, *where* $D_-$ *is the domain obtained from* $D$ *by the map* $z \mapsto -z$;
(4) $Spec(D + z_0) = Spec\ D$ *for any* $z_0 \in \mathbb{T}_P^2$.

*Remark.* In assertions (3) and (4) we are identifying $D$ with its image in the rectangle $R$, extented periodically in $\mathbb{C}$.

*Proof.* We already know that $0 \notin Spec\ D$. If $q(z)$ is a bounded solution of the equation $L_\rho q(z) = 0$, $z \in D$ for some $\rho = it$ with $t$ real, then the function $U(z) = q(\log z)|z|^{it}$ would be harmonic and bounded in every component $G$ of $\phi^{-1}(D)$ (see Proposition 0.10) and would be zero quasi-everywhere on the boundary of $G$. Hence $U \equiv 0$ and $q \equiv 0$. This proves assertion (2).

Similarly, given $\rho \in Spec\ D$, consider the harmonic function $U(z) = q(z)e^{\rho x}$. Now

$$U(z) = q(z)e^{-i\frac{2\pi}{P}x} \times e^{(\rho + i\frac{2\pi}{P})x},$$

and so the function $q_*(z) := q(z)e^{-i(2\pi/P)x}$, which is $P$-periodic and vanishes q.e. on $\partial D$, satisfies $L_{\rho*}q_* = 0$ for $\rho* = \rho + 2\pi i/P$. Also, since $\overline{U} = \overline{q}e^{\overline{\rho}x}$, we see that $\overline{\rho}$ is also an eigenvalue.

Let $D_-$ be the domain obtained from $D$ by the map $z \mapsto -z$, and $\rho_0$, $q_0$ be an eigenvalue and its eigenfunction for $D_-$. It is easy to check by changing variables that $-\rho_0$, $q(-z)$ are an eigenvalue and eigenfunction for the domain $D$.

Finally, if $q(z)$ is an eigenfunction for an eigenvalue $\rho$ in a domain $D$, then $q(z + z_0)$ also satisfies (0.15) for $D := D + z_0$. □

Theorem 0.16 (to be proved in §6) will imply that if $Spec\ D \neq \varnothing$, then it contains a least positive element. At the end of §6 we pose a conjecture which, if proved, would show that it has considerable symmetry.

**2. A necessary condition that Spec $D \neq \varnothing$.**
We begin with

*Proof of Proposition 0.10.* Let $z_0 \in D$ and $\zeta_0 \in \phi^{-1}(z_0)$. Then $T^j\zeta_0 \in \phi^{-1}(z_0)$ for all $j \in \mathbb{Z}$. Denote by $G_j$, $j \in \mathbb{Z}$, the component of $\phi^{-1}(D)$ containing $T^j\zeta_0$. Then $G_j = T^jG_0$, because of maximality and connectedness of $G_0$ and $G_j$. Similarly $G_j \cap G_l \neq \emptyset$ implies $G_j = G_l$. We have

$$\phi^{-1}(D) = \cup_{j=-\infty}^{\infty} G_j.$$

If $G_j \cap G_l = \emptyset$, $j \neq l$, then we have the case 1. Indeed, if there exists a curve $\hat{\gamma} \subset D$ homologous to a cycle $\gamma$ with $n_1 \neq 0$, then its lift $\sigma$ under $\phi^{-1}$ connects $\zeta \in G_0$ to $T^{n_1}\zeta \in G_{n_1}$, and, consequently, $G_0 = G_{n_1}$. This is a contradiction.

Now let $\hat{\gamma} \subset D$ be a curve corresponding to the cycle $\gamma$ with $\pm n_1 = k$ ($\geq 1$). We can suppose that $n_1 \geq 1$. Otherwise we replace $\hat{\gamma}$ by $-\hat{\gamma}$ with the opposite direction. Let $\zeta_0 \in \phi^{-1}(\hat{\gamma})$ and let $\sigma$ be the corresponding lift of $\hat{\gamma}$ which contains $\zeta_0$ and $T^k\zeta_0$. Denoting this component of $G$ by $G_0$, we have $T^k G_0 = G_0$.

Let $j_{\min} \geq 1$ be the least $j \geq 1$ such that for some $m \in \mathbb{Z}$ we have $G_m \cap G_{m+j} \neq \emptyset$. Then $j_{\min} \geq k$. Otherwise $G_m = G_{m+j_{\min}}$ and there exists a curve $\hat{\gamma}$ connecting $T^m\zeta_0$ to $T^{m+j_{\min}}\zeta_0$. This means that in $\mathbb{T}_P^2$ the corresponding cycle exists with $n_1 = j_{\min} < k$, that is a contradiction.

Since $T^k G_0 = G_0$, $j_{\min} = k$. Therefore $G_j \cap G_l = \emptyset$ for $0 \leq j, l \leq k-1$, $l \neq j$, i.e., $G_q$, $q = 0, 1, ..., k-1$, are disjoint. Now $T^m G_0 = T^q T^{lk} G_0 = T^q G_0$, and $T^k G_q = T^{k+q} G_0 = T^q G_0 = G_q, 1 \leq q \leq k-1$.

This proves the case 2.

$\square$

**Proposition 2.1.** *Let Spec $D \neq \varnothing$. Then $D$ is connected on spirals.*

*Proof.* Let $q$ be an eigenfunction corresponding to an eigenvalue $\rho$. Then $V^*(z) := q(\log z)e^{\rho x}$ is a well-defined (in general complex-valued) nontrivial harmonic function, vanishing quasi-everywhere on the boundary of some $T$-invariant open set $G$. The functions $\Re V^*$ and $\Im V^*$ share these properties. We can suppose that $V := \Re V^*$ is positive at some point $z_0$ of a component $G_0$ of $G$. If $V(z) \equiv 0$ we can replace $\Re V^*$ by $\Im V^*$. Otherwise $V^*(z) \equiv 0$ and hence $q(z)$ vanishes identically. This contradicts the assumption that $q$ is an eigenfunction.

Assume that $D$ is not connected on spirals. So we have $G_i =: T^{-i}G_0 \neq G_0$ for $i \neq 0$, and $G_i \cap G_l = \emptyset$ for $i, l \in \mathbb{Z}$ $i \neq l$ by Proposition 0.10.

If $G_0$ is precompact in $\mathbb{C}$, $V$ (and so $q$) vanishes identically that contradicts to the assumption $V(z_0) > 0$. Thus we may assume that each $G_i$ will have $0$ and $\infty$ in its closure (see the examples to Proposition 0.10). Let $z_0 \in G_0$, $|z_0| = 1$, and for each $j$ let $\theta_j(r)$ be the angular measure of $G_j \cap \{|z| = r\}$. Since $G_j = T^{-j}G_0$, it follows that $\theta_0(T^j r) = \theta_j(r)$. Let $G_j(n) = G_j \cap \{|z| = 2T^n\}$. By the definition of $V$ and standard estimates on harmonic measure [20, p. 112],

$$V(z_0) \leq \max_{|z|=2T^n} H(z)\omega(z_0, G_0(n), G_0) \leq CT^{n\rho}\omega(z_0, G_0(n), G_0)$$

$$\leq CT^{n\rho}e^{-\int_1^{T^n} dt/t\theta_0(t)}.$$

The integral can be computed:

$$\sum_{k=0}^{n-1} \int_{T^k}^{T^{k+1}} \frac{dt}{t\theta_0(t)} = \sum_{k=0}^{n-1} \int_1^T \frac{dt}{t\theta_k(t)} = \int_1^T \sum_{k=0}^{n-1} \frac{1}{\theta_k(t)} \frac{dt}{t}.$$

Since $G_j \cap G_l = \emptyset$ for $j \neq l$ (Proposition 0.10), $\sum_j \theta_j(r) \leq 2\pi$, so that $n^2/(2\pi) \leq \sum_0^{n-1} 1/\theta_k(t)$, and hence

$$V(z_0) \leq CT^{n\rho}e^{-C_1 n^2}$$

for some constant $C_1 > 0$. Letting $n \to \infty$, we find that $V(z_0) \leq 0$. This contradicts the assumption $V(z_0) > 0$ and proves Proposition 2.1.   $\square$

**3. Proof of Theorem 0.5.** Our approach uses the calculus of positive harmonic functions introduced by R. S. Martin [18] and popularized in the thesis of the late B. Kjellberg [13]. Martin's insight was to consider limits of ratios of the type

$$(3.1) \qquad h_n(z) = \frac{\omega(z, G \cap E_n, G)}{\omega(z_0, G \cap E_n, G)}$$

where $z_0$ is fixed in $G \cap \{|z| = 1\}$ and the $\{E_n\}$ are sets of positive capacity tending to $\infty$. We may assume, as is customary, that any function $v(z) \in \mathcal{P}$ is zero for $z$ not in $G$. Let $\Delta_\infty$ be the cluster set (in the topology of uniform convergence on compact subsets of $G$) of the functions $h_n$ which is obtained by letting the $\{E_n\}$ tend to infinity. It consists of positive harmonic functions on $G$ which are 1 at $z_0$. Notice that in general $\Delta_\infty$ need not be contained in $\mathcal{P}$, because a limit function need not vanish q.e. on $\partial G$.

Let us introduce some examples, where again $G$ and $D$ are related by (0.9). Let $G_\rho$ be the sector $\{|\arg z| < \pi/2\rho\}$. Then the cone $\mathcal{P}$ of Theorem 0.5 consists of positive multiples of the function $u_\rho(r, \theta) = r^\rho \cos(\rho\theta)$, so that $\mathcal{P}$ has dimension one and $\mathcal{P} = \mathcal{F}$.

An illuminating example of a 2–homogeneous set for which $\mathcal{F} \subset \mathcal{P}$ is the set $\Omega_0$:

$$\Omega_0 = \{y > 0\} \setminus \bigcup_{n=-\infty}^{\infty} \{y = 2^n, \ -\infty < x < 0\},$$

the upper half–plane with a sequence of horizontal rays deleted (this example provided the original motivation for this section). If $E_n \to \infty$ inside the first quadrant, the family associated to $\{E_n\}$ by (3.1) will converge to a function $u \in \Delta_\infty$ which is also in $\mathcal{F}$, and Theorem 0.5 implies that the positive multiples of $u$ span all of $\mathcal{F}$. However, if the $\{E_n\}$ tend to $-\infty$ through one of the horizontal channels, say $\mathcal{C}_k = \{x < 0, 2^k < y < 2^{k+1}\}$, then the $h_n$ converge to a function $u_k$ in $\Delta_\infty$, which will also be in $\mathcal{P}$, but which has infinite order and hence is not in $\mathcal{F}$. However, $u_k$ will be bounded outside the given channel $\mathcal{C}_k$, and so, if $u_j$ and $u_k$ are associated to two distinct channels, they will be linearly independent. So in this case $\mathcal{P}$ contains at least countably many linearly independent functions.

The remainder of this section is devoted to the proof of Theorem 0.5 .

Let us note that assertion (2) is an easy corollary of (1). Indeed, the function $H(Tz)$ is in $\mathcal{F}$ along with $H(z)$, and hence $H(Tz) = cH(z)$ with

$$c := H(Tz_0)/H(z_0) = H(Tz)/H(z)$$

Hence $c$ does not depend on $z_0$. It does not change if we replace $H$ for any $v \in \mathcal{F}$ because $v = aH$ with a constant $a$. Define $\rho(G)$ by $c = T^{\rho(G)}$, and note that $v$ satisfies (0.7). It is clear that $\rho(G)$ cannot be negative or zero because in such case $H(z)$ would be bounded and hence vanish identically.

Let us also remark that we can replace $G$ with $re^{i\psi}G$ for any $re^{i\psi}$ without loss of generality. Indeed, $G' = re^{i\psi}G$ is also $T$ homogeneous and equality $v_1(\zeta) := v(\zeta/re^{i\psi}), \zeta \in G'$ generates one-to-one maps $\mathcal{P}_G \mapsto \mathcal{P}_{G'}$ and $\mathcal{F}_G \mapsto \mathcal{F}_{G'}$, and $\rho(G') = \rho(G)$.

Next we produce $H(z)$ and prove uniqueness. Our first result, Lemma 3.3, gives a concrete way to characterize $\mathcal{F}$ in $\mathcal{P}$. We set $T_n = \{|z| = T^n\}$, and suppose as in Theorem 0.5 that $z_0 \in G$, $|z_0| = 1$, has been fixed.

For $v \in \mathcal{P}$, set

$$(3.2) \qquad \beta(v) = \limsup_{n \to \infty} \{ \max_{\zeta \in T_n} v(\zeta) \omega(z_0, T_n) \}.$$

**Lemma 3.3.** *Let $v \in \mathcal{P}$ and $\beta(v) = \infty$. Then $v$ has infinite order.*

*Proof.* Partition $T_n \cap G$ into $I_n = I_n(\varepsilon)$ and $J_n = J_n(\varepsilon)$, where $J_n = T_n \cap \{ z \in G : d(z, \partial G) < \varepsilon T^n \}$ and $I_n = (T_n \cap G) \setminus J_n$. In the sequel, we will often omit the dependency of $I_n$ and $J_n$ on $\varepsilon$.

We need the following fact:

$$(3.4) \qquad \lim_{\varepsilon \to 0} \sup_{z \in T_0} \omega(z, J_1(\varepsilon)) = 0.$$

Although this is easy to verify for most domains, in the generality in which we are working we need a more careful justification.

*Proof of (3.4).* Let us replace $G$ by $rG$ with $r$ chosen so that

$$(3.5) \qquad \omega(z_0, \partial G \cap T_1) = 0,$$

with $z_0$ a base point in $G \cap T_0$. This is possible because the function $f(t) = \omega(z_0, \partial G \cap \{ 1 < |z| < t \})$ is monotone increasing with $t$ and hence has a dense set of continuity points in any interval. Notice that this does not restrict generality because of the remarks made above at the beginning of proof Theorem 0.5.

Set $h_\varepsilon(z) := \omega(z, J_1(\varepsilon), G)$. Since $\{ h_\varepsilon, \ \varepsilon > 0 \}$ is a bounded family of harmonic functions it is a *normal* family. Hence for an arbitrary sequence $\varepsilon_j \to 0$ there exists a subsequence $\{ \varepsilon_{j'} \}$ such that $h_{\varepsilon_{j'}}$ converges uniformly on every compact subset of $G$ to a harmonic function $h(z)$.

Suppose for some sequence $\{ \varepsilon_j \}$ there exists a subsequence such that $h(z) \not\equiv 0$ and hence

$$(3.6) \qquad h(z_0) > 0$$

by minimum principle.

For every $h_\varepsilon$ we have the inequality $h_\varepsilon(z) \leq \omega(z, T_1, G), z \in G$. Thus $h$ itself satisfies the same inequality and hence $\lim_{z \to \zeta} h(z) = 0$ at any regular point $\zeta \in \partial G$, except, possibly, points of $E := \partial G \cap T_1$; i.e. q.e. on $\partial G \setminus E$. Denote as $\partial_I G$ the the set of irregular points in $\partial G$. Since $\omega(\partial_I G) = 0$, we combine these estimates with the maximum principle and (3.5) to deduce that

$$h(z_0) \leq \max\{ \zeta \in \partial G \setminus (E \cup \partial_I G) : \limsup_{z \to \zeta} h(z) \} \cdot \omega(z_0, \partial G \setminus (E \cup \partial_I G))$$

$$+ 1 \cdot \omega(z_0, \partial_I G) + 1 \cdot \omega(z_0, E) = 0 + 1 \cdot 0 + 1 \cdot 0,$$

and this contradicts (3.6). Thus $h(z) \equiv 0$ and (3.4) is proved. $\square$

With $z_0 \in G \cap \{ |z| = 1 \}$ fixed as above, choose a path $\gamma$ joining $z_0$ to $Tz_0$ in $G$, and let $\Omega$ be open with compact closure in $G$ such that $\Omega \supset \gamma$. We may

then take $\tau$ to be the Harnack constant $\tau(\gamma, \Omega)$. So if $u$ is positive and harmonic in $G \cap \{|z| < R\}$ with $R > T$ large enough so that $\Omega \subset \{|z| < R\}$, then

$$(3.7) \qquad \tau^{-1} u(z_0) \le u(T z_0) \le \tau u(z_0).$$

We now choose $\varepsilon_0 > 0$ so small in the definitions of $I_n$ and $J_n$ so that $T^n z_0 \in I_n$ and, using (3.4), so that

$$(3.8) \qquad \sup_{z \in T_0} \omega(z, J_1) \le \frac{1}{2\tau}.$$

In order to appreciate the significance of (3.2), we show that if $v \in \mathcal{P}$, then

$$(3.9) \qquad v(T^n z_0) \omega(z_0, T_n) < B v(z_0),$$

for some constant $B = B(G, z_0) < \infty$. Thus the condition $\beta(v) = \infty$, forces $v$ to grow rapidly away from the orbit of $z_0$.

To show (3.9) we first note that,

$$(3.10) \qquad \begin{aligned} \omega(z_0, J_n) &\le \omega(z_0, T_{n-1}) \sup_{\zeta \in T_{n-1}} \omega(\zeta, J_n) \\ &\le \tau \omega(z_0, T_n) \sup_{\zeta \in T_0} \omega(\zeta, J_1) \le \frac{1}{2} \omega(z_0, T_n). \end{aligned}$$

where the first inequality follows from the strong Markov property; the second one uses (3.7) and $T$–homogeneity; and the last one uses (3.8). We remark that (3.10) holds only for $n$ large, i.e. for $n \ge n_0$ where $n_0$ depends on $G$ and $z_0$ so that (3.7) can be used.

Since $\varepsilon_0 > 0$ has been fixed, it follows from Harnack's inequality on $I_n = I_n(\varepsilon_0)$ and $T$-homogeneity that there exists a constant $0 < b_0 = b(\varepsilon_0, G) < 1$ with $\min_{I_n} v(z) \ge b_0 v(T^n z_0)$ for all $n$. Thus, we deduce for $n \ge n_0$ that

$$(3.11) \qquad \begin{aligned} v(z_0) &\ge \omega(z_0, I_n) \min_{I_n} v(\zeta) \\ &\ge \frac{1}{2} \omega(z_0, T_n) \min_{I_n} v(\zeta) \\ &\ge \frac{b_0}{2} \omega(z_0, T_n) v(T^n z_0), \end{aligned}$$

where the first inequality follows by the maximum principle on the region $G \setminus I_n$; the second uses (3.10) and subadditivity of harmonic measures, i.e. the fact, which follows from the maximum principle, that

$$\omega(z_0, T_n, G \setminus T_n) \le \omega(z_0, I_n, G \setminus I_n) + \omega(z_0, J_n, G \setminus J_n);$$

and the last inequality in (3.11) uses Harnack's Inequality. Thus (3.9) is proved.

We can now finish the proof of Lemma 3.3. Take $z \in G$, $|z| \le T^n$, and set $M_n = \max_{T_n} v(\zeta)$. By assumption we have

$$\limsup M_n \omega(z_0, T_n) = \infty.$$

Given $S > 1$, (3.4) again implies that we may decrease $\varepsilon$ in the definitions of $I_n$ and $J_n$ so that

$$\sup_{T_0} \omega(z, J_1) < \frac{1}{2\tau S},$$

and so that (3.10) and (3.11) still hold, with a different constant $0 < b = b(\varepsilon, G) < 1$ instead of $b_0$. Then, for $|z| < T^n$, $z \in G$,

$$v(z) \leq M_{n+1}\omega(z, J_{n+1}) + \max_{I_{n+1}} v(\zeta)$$

$$\leq M_{n+1}\frac{1}{2\tau S} + \max_{I_{n+1}} v(\zeta)$$

$$\leq M_{n+1}\frac{1}{2\tau S} + (\tau/b)v(T^n z_0)$$

$$\leq M_{n+1}\frac{1}{2\tau S} + (2\tau/b^2)v(z_0)(\omega(z_0, T_n))^{-1}$$

where the first inequality follows from the maximum principle on $G \cap \{|z| < T^n\}$; the second one uses our choice of $\varepsilon$; the third one uses Harnack's inequality on $I_{n+1}$ (as already done just before (3.11)), as well as (3.7); and the final inequality follows (3.11). Taking the supremum over all $z$'s for which the above inequality holds, multiplying both sides by $\omega(z_0, T_n)$, and using (3.7) on $\omega(z, T_n)$, we obtain that

$$M_n\omega(z_0, T_n) \leq M_{n+1}\omega(z_0, T_{n+1})\frac{1}{2S} + Av(z_0),$$

with $A = A(\varepsilon)$, and so by iterating this inequality for $k = 1, 2, 3 \ldots$,

$$(3.12) \qquad M_n\omega(z_0, T_n) \leq \frac{1}{(2S)^k}M_{n+k}\omega(z_0, T_{n+k}) + Av(z_0)\sum_0^{k-1}\frac{1}{(2S)^j}.$$

Since $\beta(v) = \infty$, we may choose $n = n_1$ so large that

$$M_{n_1}\omega(z_0, T_{n_1}) > \frac{4SAv(z_0)}{2S - 1} > 2Av(z_0),$$

and then (3.12) implies for each $k > 1$ that

$$M_{n_1+k}\omega(z_0, T_{n_1+k}) \geq (2S)^k \left( M_{n_1}\omega(z_0, T_{n_1}) - \frac{Av(z_0)}{1 - (1/(2S))} \right) \geq Av(z_0)(2S)^k.$$

Since $S$ is arbitrary, $v$ must have infinite order. Lemma 3.3 is proved.   $\square$

We now construct functions in $\mathcal{F}$. Choose $E \subset G \cap T_0$ with $|E| > 0$ and (cf. (3.1)) let $\mathcal{H}$ consist of all normal limits of the family of functions

$$(3.13) \qquad h_n(z) = \frac{\omega(z, T^n E)}{\omega(z_0, T^n E)} \qquad (z \in G \cap \{|z| \leq T^n\} \quad n = 1, 2, 3, \ldots).$$

**Lemma 3.14.** *Let $\mathcal{H}$ be as above. Then $\mathcal{H} \subset \mathcal{F}$.*

*Proof.* We first show that for $m$ large,

$$(3.15) \qquad \sup_{T_0} \omega(\zeta, T^m E) \leq D\omega(z_0, T^m E),$$

for a constant $D > 0$ which only depends on the domain $G$. Recall that in the proof of Lemma 3.3 we created $\{I_n = I_n(\varepsilon_0)\}$, $\{J_n = J_n(\varepsilon_0)\}$ so that (3.8) holds, and recall the constants $\tau$ and $b_0$ as well. If $S_m = \sup_{\zeta \in J_0} \omega(\zeta, T^m E)$, then

$$
\begin{aligned}
S_m &\leq \sup_{I_1} \omega(z, T^m E) + \frac{1}{2\tau} \sup_{J_1} \omega(\zeta, T^m E) \\
(3.16) \qquad &\leq (\tau/b_0)\omega(z_0, T^m E) + \frac{1}{2\tau} S_{m-1} \\
&\leq (\tau/b_0)\omega(z_0, T^m E) + \frac{1}{2} S_{m-1} \frac{\omega(z_0, T^m E)}{\omega(z_0, T^{m-1} E)}
\end{aligned}
$$

where the first inequality follows from the maximum principle on $G \cap \{|z| < T\}$ and (3.8); the second one uses Harnack's inequality on $I_1$ followed by (3.7) and the definition of $S_{m-1}$; and the last line follows from (3.7) and homogeneity. We deduce from (3.16) that

$$
\frac{S_m}{\omega(z_0, T^m E)} \leq \tau/b_0 + \frac{1}{2} \frac{S_{m-1}}{\omega(z_0, T^{m-1} E)} \leq \cdots
$$

$$
\leq (\tau/b_0) \sum_{0}^{m-m_1-1} 2^{-j} + 2^{-(m-m_1)} \frac{S_{m_1}}{\omega(z_0, T^{m_1} E)} \leq 4\tau/b_0,
$$

for $m$ large enough. Harnack's inequality on $I_0$ yields that

$$
\sup_{\zeta \in I_0} \omega(\zeta, T^m E) \leq (1/b_0)\omega(z_0, T^m E),
$$

and so (3.15) holds with $D = 4\tau/b_0$.

Consider now the functions $\{h_n\}$ of (3.13), and let $|z| \leq T^n$. Then for $m > n$ ($m$ much larger than $n$),

$$
\begin{aligned}
\omega(z, T^m E) &\leq \omega(z, T_n) \sup_{\zeta \in T_n} \omega(\zeta, T^m E) \\
&\leq D\omega(z, T_n)\omega(T^n z_0, T^m E) \\
&\leq D\omega(z, T_n)\tau^n \omega(z_0, T^m E),
\end{aligned}
$$

where the first inequality follows from the maximum principle on $G \cap \{|z| < T^n\}$; the second one uses (3.15) and homogeneity; and the last one (3.7) $n$ times. Hence any normal limit $h$ of the $h_n$ must satisfy

$$
h(z) \leq D\tau^n \omega(z, T_n),
$$

for $|z| \leq T^n$, and so $h$ is locally bounded and vanishes q.e. near each finite boundary point of $G$. If we set $\tau = T^\ell$, then with the notations of (0.6), $M(T^n, h) \leq C\tau^n = C(T^n)^\ell$, so that $h \in \mathcal{F}$ (hence by Lemma 3.3, $\beta(h) < \infty$). Thus $\mathcal{F} \neq \varnothing$. $\square$

Finally, we show that $\mathcal{F}$ is one–dimensional. Let $H$ be any limit function of the family (3.1) and let $\langle H \rangle$ consist of all positive multiples of $H$. Clearly $\langle H \rangle \subset \mathcal{F}$.

**Theorem 3.17.** $\langle H \rangle = \mathcal{F}$.

*Remark.* If $G \cap \{1 < |z| < T\}$ were a Lipschitz domain, this would be a consequence of the boundary Harnack principle (cf. §5). What follows is a replacement for this principle.

The strategy to prove uniqueness is as follows: we first construct some auxiliary functions $V_\varepsilon \in \mathcal{F}$, namely for each $\varepsilon > 0$ small enough, we use the partition $\{I_n(\varepsilon), J_n(\varepsilon)\}$ that was discussed in the proof of Lemma 3.3., and then we produce $V_\varepsilon$ as a certain sublimit of the ratios (3.13) with $E = I_0$; once this is done we show that every function in $\mathcal{F}$ is comparable to the functions $V_\varepsilon$, and then we conclude using a standard argument. First we need a lemma.

**Lemma 3.18.** *There exists $\varepsilon_1 = \varepsilon_1(G) > 0$ and $K = K(G) > 1$ so that, whenever $\{I_n = I_n(\varepsilon)\}$ and $\{J_n = J_n(\varepsilon)\}$ are created as in the proof of Lemma 3.3 for any fixed $0 < \varepsilon < \varepsilon_1$, and $U$ is defined as*

$$U(z) = \lim_{k \to \infty} \frac{\omega(z, J_{m_k})}{\omega(z_0, J_{m_k})}$$

*(where $m_1 < m_2 < \ldots$ ), there is a limit function $V$ of the corresponding family $\{V_{m_k}\} = \{\omega(z, I_{m_k})/\omega(z_0, I_{m_k})\}$ with*

$$U(z) < KV(z)$$

*for all $z \in G$.*

*Proof.* With $\tau, B, D$ from (3.7), (3.9) and (3.15) choose $\varepsilon_1$ so that for $\varepsilon < \varepsilon_1$

$$\sup_{T_0} \omega(z, J_1) < \frac{1}{2BD\tau}.$$

Since $\tau, B$, and $D$ only depend on the domain $G$, also $\varepsilon_1$ only depends on $G$. Fix $m > n$ and let $|z| < T^{n-1}$. We shall analyze the inequality

$$(3.19) \qquad \omega(z, J_m) \leq \omega(z, I_n) \sup_{I_n} \omega(\zeta, J_m) + \omega(z, J_n) \sup_{J_n} \omega(\zeta, J_m),$$

which follows from the maximum principle on $G \cap \{|z| < T^n\}$.

Consider the first term on the right. With $M = \min_{I_1} \omega(\zeta, I_0)$, $0 < M < 1$, we have

$$\omega(z, I_n) \leq M^{-1}\omega(z, I_{n-1}),$$

by the maximum principle on $G \setminus (I_{n-1} \cup I_n)$ and homogeneity.

By Harnack's inequality on $I_n \cup T^{n-1}z_0$,

$$\sup_{I_n} \omega(\zeta, J_m) < A\omega(T^{n-1}z_0, J_m) \qquad (A = A(\varepsilon)),$$

so that the first term on the right side of (3.19) is at most

$$AM^{-1}\omega(z, I_{n-1})\omega(T^{n-1}z_0, J_m).$$

As for the second term,

$$\omega(z, J_n) \leq \omega(z, I_{n-1}) \sup_{I_{n-1}} \omega(\zeta, J_n)$$
$$+ \omega(z, J_{n-1}) \sup_{J_{n-1}} \omega(w, J_n)$$
$$\leq \omega(z, I_{n-1}) + \frac{1}{2BD\tau}\omega(z, J_{n-1})$$

where the first inequality follows from the maximum principle on $G \cap \{|z| < T^{n-1}\}$; and the second one our choice of $\varepsilon$. On the other hand, by (3.15), homogeneity, and (3.7), we have

$$\sup_{J_n} \omega(\zeta, J_m) \leq D\tau \omega(T^{n-1}z_0, J_m),$$

and so we deduce that the second term is at most

$$\omega(z, I_{n-1})D\tau\omega(T^{n-1}z_0, J_m) + \frac{1}{2BD\tau}\omega(z, J_{n-1})D\tau\omega(T^{n-1}z_0, J_m).$$

Combining these estimates, we obtain that

$$\omega(z, J_m) \leq (M^{-1}A + D\tau)\omega(z, I_{n-1})\omega(T^{n-1}z_0, J_m)$$
$$+ \omega(z, J_{n-1}) \cdot \frac{1}{2B}\omega(T^{n-1}z_0, J_m).$$

Divide both sides by $\omega(z_0, J_m)$, and let $m \to \infty$ appropriately. It follows that

$$U(z) \leq (M^{-1}A + D\tau)\omega(z, I_{n-1})U(T^{n-1}z_0) + \frac{1}{2B}\omega(z, J_{n-1})U(T^{n-1}z_0).$$

Note that by Lemma 3.14, $U$ belongs to $\mathcal{P}$, and (3.9) applied to $U$ refines the last estimate to

$$U(z) \leq (M^{-1}A + D\tau)B\frac{\omega(z, I_{n-1})}{\omega(z_0, I_{n-1})} + \frac{1}{2}\frac{\omega(z, J_{n-1})}{\omega(z_0, J_{n-1})},$$

where we used the fact that $U(z_0) = 1$. Now let $n - 1$ tend to infinity along an appropriate subsequence of the $\{m_k\}$. Then

$$U(z) \leq (M^{-1}A + D\tau)BV(z) + \frac{1}{2}U(z),$$

and so we may take $K = 2(M^{-1}A + D\tau)B$. Lemma 3.18 is proved.  $\square$

Suppose that for every $0 < \varepsilon < \varepsilon_1$ we construct the partition $\{I_n = I_n(\varepsilon), J_n = J_n(\varepsilon)\}$. Normal families then produce a function $U_\varepsilon$, and hence also a function $V_\varepsilon$, as in Lemma 3.18.

We next study an expression complementary to $\beta(v)$ in (3.2).

**Lemma 3.20.** *Let* $v \in \mathcal{F}$. *Then there exists* $\varepsilon_v > 0$, *so that if* $\varepsilon < \varepsilon_v$ *and* $\{I_n = I_n(\varepsilon)\}$, $\{J_n = J_n(\varepsilon)\}$ *are constructed as in Lemma 3.3, there is a constant* $A_\varepsilon > 1$ *such that*

$$\omega(z_0, I_n) \min_{I_n} v(\zeta) > v(z_0)/A_\varepsilon \qquad (n = 0, 1, 2, \dots ).$$

*Proof.* Since $\beta(v) < \infty$,

$$(3.21) \qquad\qquad S(v) = \sup_n \max_{T_n} v(\zeta)\omega(z_0, T_n) < \infty.$$

By (3.4), we may choose $\varepsilon_v$ so that if $\varepsilon < \varepsilon_v$, then

$$\sup_{T_0} \omega(z, J_1) < \frac{v(z_0)}{2\tau S(v)}.$$

By the argument which yielded (3.10), we have

$$\omega(z_0, J_n) \leq \frac{1}{2S(v)}\omega(z_0, T_n).$$

Thus,
$$v(z_0) \leq \omega(z_0, J_n) \max_{J_n} v(\zeta) + \omega(z_0, I_n) \max_{I_n} v(\zeta)$$
$$\leq \frac{v(z_0)}{2S(v)}\omega(z_0, T_n) \max_{T_n} v(\zeta) + A_\varepsilon\omega(z_0, I_n) \min_{I_n} v(\zeta)$$
$$\leq \frac{v(z_0)}{2} + A_\varepsilon\omega(z_0, I_n) \min_{I_n} v(\zeta),$$

where the first inequality follows from the maximum principle on $G \cap \{|z| < T^n\}$; the second one uses our choice of $\varepsilon$ and Harnack's inequality on $I_n$; and the last one uses (3.21). Lemma 3.20 is proved provided $A_\varepsilon$ is changed to $2A_\varepsilon$.  □

Now we show that the class $\mathcal{F} \subset \mathcal{P}$ is one-dimensional; this will prove Theorem 3.17 and thus Theorem 0.5. By choosing $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_v\}$ and letting $V_\varepsilon$ be the function constructed after Lemma 3.18, we show that any $v \in \mathcal{F}$ satisfies

$$(3.22) \qquad\qquad C^{-1}v(z) \leq V_\varepsilon(z) \leq Cv(z)$$

for all $z \in G$. Once (3.22) is established, we use a now–standard argument of Kjellberg, which we recall at the end of the proof for completeness. On "nice" domains (cf. §5) equation (3.22) would follow automatically from the boundary Harnack principle and homogeneity.

By the maximum principle, for all $z \in G$, $|z| \leq T^n$,

$$v(z) \geq \frac{\omega(z, I_n)}{\omega(z_0, I_n)}\omega(z_0, I_n) \min_{I_n} v(\zeta),$$

so, by Lemma 3.20 and the construction of $V_\varepsilon$, $v(z) \geq (v(z_0)/A_\varepsilon)V_\varepsilon(z)$. So the second inequality of (3.22) is proved.

On the other hand, if $|z| < T^n$,

$$v(z) \leq \frac{\omega(z, I_n)}{\omega(z_0, I_n)} \max_{I_n} v(\zeta) \omega(z_0, I_n) + \frac{\omega(z, J_n)}{\omega(z_0, J_n)} \max_{J_n} v(\zeta) \omega(z_0, J_n)$$

$$\leq \left[ \frac{\omega(z, I_n)}{\omega(z_0, I_n)} + \frac{\omega(z, J_n)}{\omega(z_0, J_n)} \right] \max_{T_n} v(\zeta) \omega(z_0, T_n),$$

where both inequalities follow from the maximum principle on $G \cap \{|z| < T^n\}$. Since $v \in \mathcal{F}$, Lemma 3.3 implies that $\limsup_{n\to\infty} \max_{T_n} v(\zeta) \omega(z_0, T_n) = \beta(v) < \infty$. Hence, by Lemma 3.18, $v(z) \leq CV_\varepsilon(z)$, and (3.22) follows.

We now repeat Kjellberg's argument. Suppose $v_1, v_2 \in \mathcal{F}$. Two applications of (3.22) imply that $v_1$ and $v_2$ i must also be comparable. Set

$$m = \inf_G \frac{v_2(z)}{v_1(z)} \in [C^{-1}, C].$$

We claim that $v_2 - mv_1 \equiv 0$. Assume the contrary. Then $w = v_2 - mv_1 \in \mathcal{F}$, and by (3.22) $w(z)/v_1(z) > 1/C_1$ for every $z \in G$, for some other constant $C_1 > 1$. But

$$\inf_{z \in G} \frac{w(z)}{v_1(z)} = m - m = 0.$$

This is a contradiction. Therefore $w = 0$ and $v_2 \equiv mv_1$. So $\mathcal{F}$ consists of positive multiples of a single function.

**4. $\rho(G)$ as an order of growth and decay of harmonic functions in $G$.** Let $H$ be the unique function in $\mathcal{F}$ as determined by Theorem 0.5. Set

$$M(r, H) = \sup_\theta \{H(re^{i\theta}) : z = re^{i\theta} \in D\}.$$

**Proposition 4.1.** *The function $H$ is related to $\rho(G)$ by*

$$(4.2) \qquad \rho(G) = \lim_{r\to\infty} \frac{\log M(r, H)}{\log r} = \lim_{r\to 0} \frac{\log M(r, H)}{\log r},$$

*where the limits exist and are positive.*

*Proof.* From (0.7) we obtain that $M(Tr, H) = T^{\rho(G)} M(r, H)$, which implies

$$\log M(T^n r, H) = n\rho(G) \log T + \log M(r, H).$$

This means that

$$\rho(G) = \lim_{n\to\infty} \frac{\log M(T^n, H)}{n \log T} = \lim_{n\to-\infty} \frac{\log M(T^n, H)}{n \log T}.$$

Note that $M(r, H)$ is increasing with $r$, by the maximum principle. This and monotonicity of $\log r$ imply (4.2). From Theorem 0.5 $\rho(G) > 0$. $\square$

The constant $\rho(G)$ is also intimately related to the decay of harmonic measure. Recall that $\omega(z, T_n)$ is the harmonic measure of $T_n := \{z : |z| = T^n\}$ with respect to $G$.

**Proposition 4.3.** *For each $z \in G$ there corresponds $C = C(z) > 1$ such that*

(4.4) $$(1/C)T^{-n\rho(G)} \leq \omega(z, T_n) \leq CT^{-n\rho(G)} \quad (n \geq 1).$$

*In particular, for each $z \in G$,*

(4.5) $$\lim_{r \to \infty} \frac{\log \omega(z, \{|\zeta| = r\})}{\log(1/r)} = \lim_{n \to \infty} \frac{\log \omega(z, T_n)}{\log T^{-n}} = \rho(G)$$

*Proof.* Let $H \in \mathcal{F}$ be from Theorem 0.5 with $H(z_0) = 1$. Then (3.9) yields that

$$\omega(z_0, T_n) \leq B[H(T^n z_0)]^{-1} = BT^{-n\rho(G)},$$

which, using Harnack's inequality to compare $\omega(z, T_n)$ to $\omega(z_0, T_n)$, yields the right-hand estimate of (4.4), while Lemma 3.20 shows that

$$\omega(z_0, T_n)H(T^n z_0) \geq \omega(z_0, I_n)\min_{I_n} H(\zeta) \geq A,$$

leading to the reverse inequality. So (4.4) is proved.

The second equality in (4.5) follows obviously from (4.4). The first one follows from the second one and monotonicity of $\omega(z, \{|\zeta| = r\})$ and $\log(1/r)$ in $r$. $\square$

**4.6.** The set-function $\rho(G)$ has a classical interpretation in the situation that $G$ is simply-connected and $T$-invariant. Let $I_0$ be the interval of $G \cap T_0$ that separates zero from infinity in $G$ and set $I_1 = TI_0$. Since $G$ is simply-connected, $I_0$ divides $G$ into two components, as does $I_1$. Let $R(I_0, I_1)$ be the quadrilateral which is the component of $G$ having $I_0$ and $I_1$ on its boundary, and let $\mod R(I_0, I_1)$ be its conformal modulus. Namely $\mod R(I_0, I_1)$ is the length $L$ of the unique rectangle $Q = \{z = x + iy : 0 < x < L, 0 < y < 1\}$, which can be obtained by mapping $R(I_0, I_1)$ to $Q$ conformally in such a way that $I_0$ and $I_1$ are mapped to the two vertical sides respectively.

Let us make the following construction. Denote $G_0 := R(I_0, I_1)$ and set $G_n := T^n R(I_0, I_1) = R(I_n, I_{n+1})$, $I_n = T^n I_0$ $n \in \mathbb{Z}$. Replace every $G_{2k-1}$, $k \in \mathbb{Z}$ by $G_{2k}^*$, the quadrilateral that is symmetric to $G_{2k}$ with respect to the arc $I_{2k}$. We obtain a new $T^2$-homogeneous domain $G^S$, or two-sheeted Riemann surface, because a point can be covered by some $G_{2k}$ and possibly by some $G_{2m}^*$.

We do not develop our theory for Riemann surfaces, so we confine ourselves to domains $G$ with a *separating circle*: there exists a circle that intersects $G$ on one arc. Without loss of generality take that to be $I_0$. Then $G_n$ and $G_{n-1}^*$ have the only common arc $I_n = T^n I_0 \in G$. So if $G$ has a separating circle $G^S$ is a plane domain and vice versa.

**Theorem 4.7.** *If $G$ is a domain with a separating circle*

(4.8) $$\rho(G^S) = (\pi/P) \mod R(I_0, I_1).$$

*Proof.* Let $w = f(z)$ map $R := G_0$ conformally to the rectangle $(0, c) \times (0, \pi)$ (for a unique $c$) so that $I_0$ and $I_1$ correspond respectively to the vertical sides.

By reflection, we may extend $f$ to map $R^* = G^*_{-1}$ to the rectangle $(-c, 0) \times (0, \pi)$, and in the same way extend $f$ to map $G^S$ to the strip $\{0 < v < \pi\}$ so that

$$f(T^2 z) = 2c + f(z), \ \Re f(T^2 z) = 2c + \Re f(z),$$

On iterating, we have that $\Re f(T^{2n} z) = 2cn + \Re f(z)$.

The function $H(z) := \Im e^{f(z)} = e^{\Re f(z)} \sin \Im f(z)$ is positive and harmonic within $G^S$ and equal to zero at every regular point of boundary. One can check that it has a finite order. By (4.2)

$$\rho(G^S) = \lim_{n \to \infty} \frac{2cn + O(1)}{2n \log T} = \lim_{n \to \infty} \frac{2cn + O(1)}{2nP} = \frac{c}{P} = \frac{\pi}{P} \cdot \mod R(I_0, I_1).$$

$\square$

**Corollary 4.9.** *If $G$ is a T-homogeneous domain with a separating circle which is also symmetric with respect to reflection in this circle, then*

$$\rho(G) = (\pi/P) \cdot \mod R(I_0, I_1).$$

The set-function $\rho(G)$ has several other interpretations in the situation that $G$ is simply-connected and $T$-invariant (see Lemma 6.4 and Theorem 6.6 of [19]). Thus Proposition 4.3 can also be formulated in terms of the growth of the distance in the hyperbolic metric of $G$ between $T_0$ and $T_n$, as well as the growth of the extremal distance between $T_0$ and $T_n$. These latter objects make sense also for the non-simply-connected domains $G$ (because we assume the capacity of the boundary to be non-zero), although explicit computation may be more difficult.

Here we discuss one such result related to [19].

**Proposition 4.10.** *Let $G$ be simply connected and $T$-invariant for some $T > 1$. Let $I_0$ be an arc of $G \cap \{|z| = 1\}$ which separates zero from $\infty$ in $G$ and let $I_n = T^n I_0$. Then*

$$\rho(G) = \lim_{n \to \infty} \frac{\pi}{\log T} \frac{d_G(I_0, I_n)}{n}.$$

*where $d_G(I_0, I_n)$ is the extremal distance between the two crosscuts $I_0$ and $I_n$ in the quadrilateral formed by them in $G$.*

The same formula holds if one uses all the arcs $T_0 = G \cap \{|z| = 1\}$ and $T_n = T^n T_0$, however the proof is much more delicate and can be deduced from the proof of Claim 6.7 of [19]. The problem is that in general, with the notations of the proof of Proposition 4.10 below, $\psi^{-1}(T_0)$ can be quite pathological, so that both 0 and $\infty$ are in its closure.

*Proof of Proposition 4.10.* As constructed in Lemma 6.4 of [19] let $\psi$ be a conformal map of the upper half-plane $\mathbb{H}$ onto $G$ such that

$$\psi(tz) = T\psi(z)$$

for some $t > 1$. Let $J_0 = \psi^{-1}(I_0)$ and $J_n = \psi^{-1}(I_n) = t^n J_0$. By properties of conformal maps, $J_0$ is a Jordan arc in $\mathbb{H}$ with two end-points $a, b \in \mathbb{R} \setminus \{0\}$. Therefore $M = \max\{|z| : z \in J_0\}$ and $m = \min\{|z| : z \in J_0\}$ are well defined

with $0 < m \le M < \infty$. Let $C_r$ be the arc $\{|z| = r\} \cap \mathbb{H}$. By conformal invariance, $d_G(I_0, I_n) = d_{\mathbb{H}}(J_0, J_n)$, so for $n$ sufficiently large,

$$\frac{1}{\pi}(n \log t - \log \frac{M}{m}) \le d_G(I_0, I_n) \le \frac{1}{\pi}(n \log t + \log \frac{M}{m}).$$

Hence

$$\lim_{n \to \infty} \pi \frac{d_G(I_0, I_n)}{n} = \log t.$$

The function $H$ of Theorem 0.5 is a positive multiple of $h(z) = \Im\psi^{-1}(z)$, and $h(Tz) = th(z)$. Thus (0.7) yields that

$$\rho(G) = \frac{\log t}{\log T}.$$

$\square$

**5. A sufficient condition that $\mathcal{F} = \mathcal{P}$.** We saw at the beginning of §3 that in general $\mathcal{F} \subset \mathcal{P}$, with strict inclusion possible. Suppose, however, that $G$ is a $T$– homogeneous Lipschitz domain or, more generally, the boundary Harnack principle holds on $T_0 \cap G$: i. e., there is $\varepsilon > 0$ and a constant $C > 1$ such that for every pair of positive harmonic functions $u$ and $v$, locally bounded and vanishing q. e. near each point of $\partial G \cap \{1 - \varepsilon < |z| < 1 + \varepsilon\}$, we have

$$\frac{u(\zeta)}{v(\zeta)} \le C \frac{u(z_0)}{v(z_0)}$$

for all $\zeta \in T_0 \cap G$ and for some $z_0 \in T_0 \cap G$. For instance, this happens if $T_0$ consists of finitely many arcs and in a neighborhood of each end-point of these arcs the boundary of $G$ is a (possibly rotated) graph of a Lipschitz function (cf. [5, p. 178]). By homogeneity, the same constant $C$ works on each $T_n \cap G$. So, given a function $v \in \mathcal{P}$, if $H$ is the Martin function constructed in Theorem 0.5, let $z_0$ be a point on $T_0 \cap G$ where $M(1, H)$ (as defined in Proposition 4.1) is attained. Then for all $z \in T_n \cap G$,

$$v(z) \le C \frac{H(z)}{H(T^n z_0)} v(T^n z_0) \le Cv(T^n z_0) \le \frac{B}{\omega(z_0, T_n)}$$

where we used (3.9) for the last inequality. Thus $\beta(v) < \infty$ (recall (3.2)) and so $v \in \mathcal{F}$.

**6. Special properties of the spectrum.** Let $G$ be a $T$–homogeneous domain, and $\rho(G)$ be associated to $G$ as in Theorem 0.5. (For example, as we noted after the statement of Theorem 0.19, if $G = \{z; |\arg z| < \theta\}$, then $\rho(G) = \pi/2\theta$.) We first show that $\rho(G)$ is a (strictly) monotonic set function.

**Theorem 6.1.** *Let $G_1 \subset G_2$ be $T$–homogeneous domains such that $E = G_2 \backslash G_1$ has positive capacity. Then $\rho(G_1) > \rho(G_2)$.*

*Proof.* Since $TE = E$, it follows that $\mathrm{Cap}(E \cap \{1 \le |z| < T\}) > 0$. Without loss of generality, we may choose a compact set $K \subset E \cap \{1 < |z| < T\}$ of positive

capacity such that $K \subset\subset G_2 \cap \{1 < |z| < T\}$; if $E \subset \{|z| = 1\}$, we replace $G_2, G_1$ by $\lambda G_2, \lambda G_1$ for $\lambda$ close to 1.

Let $T_n = \{|z| = T^n\}$ and $z_0 \in G_1 \cap G_2$ with $|z_0| = 1$.

Recall that for a $T$–homogeneous domain $G$ and a compact set $K \subset G$ the harmonic measure $\omega(z, K, G)$ satisfies the equality $\omega(Tz, TK, G) = \omega(z, K, G)$.

Now fix $n > 1$ and for $0 \le j \le n - 1$ put

$$(6.2) \qquad m_j = \inf_{z \in T^j K} \omega(z, T_n, G_2).$$

By the maximum principle,

$$(6.3) \quad \omega(z_0, T_n, G_2) \ge \omega\left(z_0, T_n, G_2 \backslash \bigcup_0^{n-1} T^j K\right) + \sum_0^{n-1} m_j \omega\left(z_0, T^j K, \ G_2 \backslash \bigcup_{\ell=1}^{n-1} T^\ell K\right),$$

where $m_j$ is from (6.2) and since $K \subset\subset G_2$, the Harnack inequality yields $m_j \ge C\omega(T^j z_0, T_n, G_2)$ for some $C = C(G_2) > 0$. Hence we have from (3.15) (for a different $C$) that

$$(6.4) \qquad m_j \ge C \sup_{\zeta \in T_j} \omega(\zeta, T_n, G_2).$$

Using the ideas of §3, let $I_j = \{z \in T_j \cap G_1, \, d(z, \partial G_1) > \varepsilon T^j\}$, where $0 < \varepsilon < d(z_0, \partial G_1)$. Then

$$\omega\left(z_0, T^j K, \ G_2 \backslash \bigcup_{\ell=0}^{n-1} T^\ell K\right) \ge \omega(z_0, I_j, G_1) \inf_{\zeta \in I_j} \omega\left(\zeta, T^j K, G_2 \backslash \bigcup_{\ell=1}^{n-1} T^\ell K\right)$$

$$\ge \omega(z_0, I_j, G_1) \inf_{\zeta \in I_j} \omega\left(\zeta, T^j K, G_2 \backslash \bigcup_{-\infty}^{\infty} T^\ell K\right) = A_j \omega(z_0, I_j, G_1),$$

where, $A_j = A_0$ is independent of $j$. Since $K \cap T_0 = \varnothing$, $A_0 > 0$.

The argument which gave (3.10) shows that to $\varepsilon < \varepsilon_0$ corresponds $n_0 = n_0(z_0)$ so that

$$(6.5) \qquad \omega(z_0, I_j, G_1) \ge \frac{1}{2}\omega(z_0, T_j, G_1) \quad (j > n_0).$$

Hence (6.4) and (6.5) imply that when $j > n_0$, each term in the sum in (6.3) is greater than

$$\frac{1}{2} C A_0 \omega(z_0, T_j, G_1) \sup_{\zeta \in T_j} \omega(\zeta, T_n, G_1) \ge C_1 \omega(z_0, T_n, G_1),$$

if we take $C_1 = C A_0 / 2$. This transforms (6.3) to

$$\omega(z_0, T_n, G_2) \ge (1 + (n - n_0) C_1)\omega(z_0, T_n, G_1),$$

and so Theorem 6.1 follows from (4.4). .   $\square$

We can now prove Theorem 0.16.

*Proof of Theorem 0.16.* Let $D \subset \mathbb{T}_P^2$ be connected on spirals and let $G$ be a component of $\phi^{-1}(D)$. Proposition 0.10 implies that $G$ is a $T^k$-homogeneous domain, for some $k \in \mathbb{N}$. Thus equation (0.7) from Theorem 0.5 (with $T$ replaced by $T^k$) implies that $\rho_0 := \rho(G)$ is a positive point of *Spec D* with eigenfunction $q(z) = H(e^z)e^{-\rho(G)\Re z}$. Together with Proposition 2.1 this proves Theorem 0.16, (1).

Let $\rho^* := \rho(D)$ be the minimal positive eigenvalue of the boundary problem (0.15). It exists because the set of positive eigenvalues is not empty as we have just shown, is discrete without any finite point of condensation and does not contain zero (Propositions 1.36, 1.37). Now we are going to prove that $\rho(G) = \rho(D)$ and this proves Theorem 0.16, (2).

Since $\rho^* \leq \rho_0$ we must prove $\rho^* \geq \rho_0$.

Denote by $q^*(z)$ the eigenfunction corresponding to $\rho^*$. We may assume that $q^*(z_1) = 1$ for some $z_1 \in D$.

Let $G$ be a component of $\phi^{-1}(D)$, and let $v^*(z) = q^*(\log z)|z|^\rho$, $z \in G$, so that $v^*$ and $q^*$ are related by (0.8). Also, let $G^* \subset G$ be the component of $\{v^*(z) > 0\}$ which contains the preimage of $z_1$. Then $v^*$ is positive harmonic in $G^*$ and vanishes quasi-everywhere on the boundary. By Theorem 6.1 $\rho^* \geq \rho_0$, and this establishes the remaining assertion of Theorem 0.16.   $\square$

*Proof of Theorem 0.17.* This follows directly from Theorem 6.1 and the equality $\rho(G) = \rho(D)$. Indeed, $D_1 \subset D_2 \implies G_1 \subset G_2$ for an arbitrary $G_2 \in \phi^{-1}(D_2)$ and $G_k$, $k = 1, 2$ corresponding to $D_k$, $k = 1, 2$ by (0.9). The supposition $\text{Cap}(D_2 \setminus D_1) > 0$ implies $\text{Cap}(G_2 \setminus G_1) > 0$ since analytic maps preserve positive capacity.   $\square$

**Proposition 6.6.** *Let $D_n \uparrow D$ be a sequence of domains in $\mathbb{T}_P^2$ and $q_n$, $n = 1, 2, \ldots$ are the corresponding normalized by condition $q_n(z_0) = 1, z_0 \in D$ solutions of the problem (0.15).*

*Then $\rho(D_n) \downarrow \rho(D)$ and $q_n \to q$ uniformly in any compact set $K \subset D$ and quasi-everywhere on $\partial D$.*

*Proof.* Since each $D_n$ can be approximated from inside by smooth domains (see, e.g. [12]), we can suppose that each $D_n$ is smooth.

Set $\rho^* := \lim_{n \to \infty} \rho(D_n)$. This limit exists because the sequence $\rho(D_n)$ decreases monotonically and is bounded below by $\rho(D)$. Consider the sequence $\{H_n\}$ of functions

$$H_n(z) := q_n(\log z)|z|^{\rho(D_n)}.$$

Each $H_n$ is positive harmonic the domain $G_n$ (which corresponds to $D_n$ by (0.9)), vanishes on the boundary, and the sequence $\{H_n\}$ is compact. Consider any convergent subsequence $H_k \to H = q^*(\log z)|z|^{\rho^*}$.

Since the $\{q_n\}$ are normalized and converge uniformly on compacta, we have $q^*(z_0) = 1$. In addition, since $\{H_k\}$ (where each $H_k$ is extended to be zero outside $G_k$) is a sequence of subharmonic functions in $\mathbb{C}$, the function $H$ is zero quasi-everywhere on $\partial G$ by the theorem of H. Cartan ([10], Chapter 7). It is also positive harmonic in $G$. By Theorems 0.5 and 0.16 $\rho^* = \rho(D)$ and $q^* = q$.   $\square$

Let $G$ be a component of $\phi^{-1}(D)$ (see, (0.9)). The point $0 \in \partial G$ plays a role analogous to that of $\infty$, and this provides information which will supplement Proposition 1.37.

**Proposition 6.7.** *Let $D$ be a domain in $\mathbb{T}_P^2$ which is connected on spirals so that $Spec\, D$ is non-empty. Then the largest negative point in $Spec\, D$ is $-\rho(D)$.*

*Proof.* Suppose first that $D$ is smooth enough and hence $\mathcal{F} = \mathcal{P}$ (see §5). Let $G$ be related to $D$ as above and let $H^+$ be the function provided by Theorem 0.5. If $G(z, w)$ is the usual Green function for the domain $G$ and $z_0 \in G$ is a given base point, then

$$(6.8) \qquad H^+(z) = \lim_{n \to +\infty} \frac{G(z, T^n z_0)}{G(z_0, T^n z_0)}.$$

Now recall that in Proposition 1.37 we defined the domain $D_-$ as the $\mathbb{T}_P^2$-domain obtained from $D$ via the map $z \mapsto -z$. This corresponds to changing the plane-domain $G$ to $G_-$ via the map $z \mapsto 1/z$. Define $H^+_{G_-}$ as above for the domain $G_-$. Then letting $H^-(z) = H^+_{G_-}(1/z)$ we obtain from (6.8) that

$$H^-(z) = \lim_{n \to +\infty} \frac{G(z, T^{-n} z_0)}{G(z_0, T^{-n} z_0)}.$$

Let $\rho = \rho(D) > 0$ and $\sigma = \rho(D_-) > 0$. These constants give

$$H^+(Tz) = T^\rho H^+(z) \qquad H^-(T^{-1} z) = T^\sigma H^-(z).$$

But by definition

$$H^+(Tz) = \lim_{n \to +\infty} \frac{G(Tz, T^n z_0)}{G(z_0, T^n z_0)} \frac{G(z, T^n z_0)}{G(z, T^n z_0)},$$

so that

$$T^\rho = \lim_{n \to +\infty} \frac{G(Tz, T^n z_0)}{G(z, T^n z_0)}.$$

Likewise we find that

$$T^\sigma = \lim_{n \to +\infty} \frac{G(T^{-1} z, T^{-n} z_0)}{G(z, T^{-n} z_0)}.$$

Notice that $z$ can be chosen arbitrarily. In particular if we let $z = z_0$:

$$T^\sigma = \lim_{n \to +\infty} \frac{G(T^{-1} z_0, T^{-n} z_0)}{G(z_0, T^{-n} z_0)} = \lim_{n \to +\infty} \frac{G(T^n z_0, T z_0)}{G(T^n z_0, z_0)}$$

$$= \lim_{n \to +\infty} \frac{G(T z_0, T^n z_0)}{G(z_0, T^n z_0)} = T^\rho :$$

$\rho = \sigma$.

Let $D$ be an arbitrary domain connected on spirals. There exists a sequence $D_n$ of domains with smooth boundary such that $D_n \uparrow D$. Then the assertion of Proposition 6.7 for any domain $D$ follows by Proposition 6.6 that is also a corollary of Theorem 0.5. $\square$

Proposition 6.7 shows that $\rho(D_-) = \rho(D)$. This had been conjectured by V. Matsaev in a more general form, namely, that $Spec(D) = Spec(D_-)$. In the generality of this paper, this question seems to be open at the moment.

**7. Green function and Dirichlet problem.** The following theorem defines the Green function corresponding to $L_\rho$ and gives its properties.

**Theorem 7.1.** *Let $D \subset \mathbb{T}_P^2$ be a domain, $\rho \notin Spec\ D$ be given and $L_\rho$ as in (0.15). Then there exists a fundamental solution $g_{L_\rho}(z, \zeta, D)$ such that*

$$(7.2) \qquad\qquad\qquad L_\rho g_{L_\rho}(z, \zeta) = \delta_\zeta(z)$$

$$(7.3) \qquad\qquad\qquad \lim_{z \to z'} g_{L\rho}(z, \zeta, D) = 0$$

*when $\zeta \in D$ and $z' \in \partial D$ is a regular point in the sense of potential theory, and the limit is uniform for $\zeta \in K$, $K$ compact in $D$, and*

$$(7.4) \qquad\qquad g_{L_\rho}(z, \zeta, D) \le 0 \qquad (z, \zeta \in D),\ 0 < \rho < \rho(D).$$

*Proof.* From the Fredholm theorem we obtain that the equation

$$\mathcal{G}_D(\rho) := I + 2\rho G_D^1 + \rho^2 G_D = f$$

has a unique solution $q \in L^2(\mathbb{T}_P^2)$ for $\rho \notin Spec$ and $f \in L^2(\mathbb{T}_P^2)$. Take $f := g(z, \zeta, D)$, where $g$ is the Green function of the Laplace operator and $g(\cdot, \zeta, D) \in L^2(\mathbb{T}_P^2)$ by Proposition 1.12 uniformly with respect to $\zeta$. Recall that $G_D^1 q(z)$, $G_D q(z) \to 0$ as $z \to z_0$ for every regular point of $\partial D$ (see (1.31)).

Thus $q$ satisfies (7.3). One can show, following the proof of Proposition 1.28 , that (7.2) is satisfied too.

Let us prove (7.4). If $g_{L_\rho}$, which is a real function for a real $\rho$, were to change sign in $D$, consider a component

$$D^- = \{z \in \mathbb{T}_P^2; g_{L_\rho}(z, \zeta, D) < 0\}.$$

Then $D \setminus D^-$ has positive capacity, and $D^- \subset D$ is a domain in which $L_\rho g_{L_\rho} = 0$. According to Theorem 0.17, $\rho(D^-) > \rho(D)$. Since $\rho < \rho(D^-)$, we have from the definition of $\rho(D^-)$ that $g_{L_\rho} \equiv 0$ in $D^-$. Note that $g_{L_\rho}$ can not be zero in $D$ without changing sign in $D$ because of the maximum principle for the harmonic function $H(z) = g_{L_\rho}(z)e^{\rho z}$.

*Remark.* The explicit form of $g_{L_\rho}$ is given by the expression

$$g_{L_\rho}(z, \zeta) = \sum_{k=-\infty}^{\infty} g(e^z, e^{\zeta+kP}, G)e^{\rho(\xi+kP)}e^{-\rho x} \ (z = x + iy, \zeta = \xi + i\eta).$$

where $g(\cdot, \cdot, G)$ is the Green function of the initial domain $G \in \phi^{-1}(D)$ related to $D$ by (0.9). One can prove the convergence of this series for $0 < \rho < \rho(D)$ and $z \ne \zeta$, using (4.4).

Now consider the Dirichlet problem

$$(7.5) \qquad\qquad\qquad L_\rho q = 0,\ q|_{\partial D} = f$$

where $f$ is continuous in $\partial D$, $D$ is regular domain (i.e., sufficiently smooth so that $q(z) \to f(\zeta)$ while $z \to \zeta\ \forall \zeta \in \partial D$), $\rho \notin Spec\ D$.

**Proposition 7.6.** *If $0 < \rho \notin Spec\ D$, then the solution $q(z, f, D)$ to the problem (7.5) is unique. If $0 < \rho < \rho(D)$ and $f \geq 0$ on $\partial D$, then $q \geq 0$ in $D$.*

*Proof.* If $q_1$ and $q_2$ solve (7.5), then $q_1 - q_2$ would be the unique solution to the homogeneous problem (0.15), and so by our assumption on $\rho$, $q_1 \equiv q_2$.

Next, let $q$ solve (7.5) where $f \geq 0$ on $\partial D$, and suppose $q(z_0) < 0$. Then $q = 0$ on the boundary of a connected component of the open set $D^- = \{z;\ q(z) < 0\}$. But $D^- \subset D$, so $\rho(D^-) \geq \rho(D) > \rho$. Thus $q \equiv 0$ in $D^-$, which is a contradiction. $\quad\square$

Let $f$ be upper semicontinuous on $\partial D$, and consider a sequence of continuous functions $f_n \downarrow f$. The corresponding sequence $q_n = q(z, f_n, D)$ (using $f_n$ in (7.5)) converges monotonically, and defines a unique solution to (7.5) for upper semicontinuous $f$. The same holds for lower semicontinuous functions. Since every measurable function can be represented as a sum of functions of these two types, the solution is defined and unique for all mesurable functions. It can be equal to $\infty$ or $-\infty$. This is a generalized solution in the sense of Wiener to the problem (7.5); see, for example, [10].

## 8. Subfunctions with respect to the operator $L_\rho$.

An upper semicontinuous function on $\mathbb{T}_P^2$ is called an $L_\rho$–*subfunction* if, $L_\rho v \geq 0$ in the sense of distributions. If both $v$ and $-v$ are $L_\rho$–s.f., we call $v$ an $L_\rho$–*function*. The theory of $L_\rho$-subfunctions parallels that of subharmonic functions because of

**Proposition 8.1.** *A function $v$ is an $L_\rho$–s.f. iff the function*

$$V(z) = v(\log z)|z|^\rho$$

*is subharmonic in $\mathbb{C}$ and satisfy*

(8.2) $$V(e^P z)e^{-\rho P} = V(z).$$

*Proof.* That $V$ is well–defined, upper–semicontinuous and satisfying (8.2) follows from properties $V$ inherits from $v$.

We claim that $\Delta V \geq 0$ in $\mathcal{D}'(\mathbb{C})$. If $\Psi \in \mathcal{D}(\mathbb{C}\backslash 0)$, then $\Psi$ may be written as $\Psi(z) = |z|^{-\rho}\psi(z)$, with $\Psi \in \mathcal{D}(\mathbb{C}\backslash 0)$, and we may suppose that the support of $\Psi$ is contained in a sector $\Delta(\alpha, \beta, R, P) = \{re^{i\varphi};\ \varphi \in (\alpha, \beta),\ |\log(R/r)| < P\}$. Let

$$r = e^x,\ \varphi = y,\ z = x + iy,\ \hat{\psi}(z) = \psi(e^z), v(z) = e^{-\rho x}V(e^z),$$
$$\Delta(\psi(z)|z|^{-\rho}) = L_\rho^* \hat{\psi}(\log z)e^{-(\rho-2)x},$$

where $L_\rho^* = L_{-\rho}$ is symmetric to $L_\rho$ .

Then

$$\langle \Delta\psi(\cdot)| \cdot |^{-\rho}, V\rangle = \int L_\rho^* \hat{\psi}(z)v(z)dxdy$$
$$= \langle \hat{\psi}, L_\rho v\rangle_{\mathbb{T}_P^2} \geq 0,$$

since $L_\rho v \geq 0$ in $D'(\mathbb{T}_P^2)$. Thus $V$ is subharmonic in $\mathbb{C}\backslash 0$, and since $V$ is bounded near 0, $V$ extends to be subharmonic at the origin.

The sufficiency follows in the same way. $\quad\square$

**Corollary 8.3.**  *The following holds*

    *(1) If $v_1, \ldots, v_k$ are $L_\rho - s.f$ then $\max\limits_i v_i$ is an $L_\rho$–s.f.;*

    *(2) The set of $L_\rho$ subfunctions form a positive cone;*

    *(3) The set of $L_\rho$ subfunctions are closed under translation of coordinates;*

    *(4) If $d\mu \geq 0$ and $v$ is $L_\rho$–s.f., then*

$$\int_{\mathbb{T}_P^2} v(z - \zeta) d\mu(\zeta)$$

    *is an $L_\rho$–s.f.*

Let $v$ be an $L_\rho$–s.f. Consider the $C^\infty-$ function in $\mathbb{C} \setminus 0$

$$V_\varepsilon(z, v) = |z|^{-2} \int \alpha_\varepsilon(\zeta/z) v(\log \zeta) |\zeta|^\rho d\zeta$$

where $\alpha_\varepsilon \in \mathcal{D}(\mathbb{C})$, $\alpha_\varepsilon \geq 0$, $\alpha_\varepsilon(z) = 0$ for $|z - 1| > \varepsilon$ and $\int\limits_{|\zeta - 1| < \varepsilon} \alpha_\varepsilon(\zeta) d\zeta = 1$. It is easy to verify that $V_\varepsilon$ is subharmonic and satisfies (8.2). Thus

(8.4)
$$\mathcal{M}_\varepsilon(z, \cdot) := e^{-\rho x} V_\varepsilon(e^z, \cdot)$$

is an $L_\rho$–s.f., and a straightforward computation shows that

(8.5)
$$\lim_{\varepsilon \to 0} \mathcal{M}_\varepsilon(z, v) = v(z).$$

This will yield the first assertion of

**Proposition 8.6.**  (1) *Every $L_\rho$–s.f. is a decreasing limit of a sequence of infinitely differentiable $L_\rho$–s.f.'s.*

    (2) *A non-zero $L_\rho$–s.f. $v$ cannot attain a local non-positive maximum in a domain $D \subset \mathbb{T}_P^2$.*

*Proof.*

    Now we prove (2). Let $v(z_{\max}) := -c \ (\leq 0)$ be the maximal value of $v(z)$. We apply the maximum principle to the subharmonic function $V$ associated to $v_1(z) := v(z) + c$ in Proposition 8.1 and obtain that $V(z) \leq 0$ and $V(z_{\max}) = 0$, hence $V(z) \equiv 0$. Thus $v(z) \equiv -c$. If $c > 0$, then $L_\rho(-c) < 0$ that is a contradiction.  $\square$

**8.7. Green Potential.**  Let $D \subset \mathbb{T}_P^2$, $0 < \rho < \rho(D)$, $v$ an $L_\rho$–s.f., and let $g_{L_\rho}$ be the Green function of $L_\rho$, cf. §7. Consider the *Green potential* of a measure $\nu$ on $D$:

$$\Pi(z, \nu) = \int_D g_{L_\rho}(z, \zeta, D) d\nu(\zeta).$$

**Proposition 8.8.**  $\Pi(z, \nu)$ *is an $L_\rho$–s.f. in $D$ with*

$$L_\rho \Pi(\cdot, \nu) = \nu \qquad in \qquad \mathcal{D}'(\mathbb{T}_P^2).$$

*Moreover, if $supp(\nu) \subset\subset D$ or if $\nu$ has bounded density in a neigborhood of a regular boundary point $z_0 \in \partial D$, then*

$$\lim_{D \ni z \to z_0} \Pi(z, \nu) = 0.$$

*Proof.* The function $\Pi(z, \nu)$, being the potential of a negative kernel, is upper semicontinuous. Next, let $\Psi > 0 \in \mathcal{D}(\mathbb{T}_P^2)$. By Theorem 7.1, we have

$$\langle L_\rho \Pi, \psi \rangle = \int L^* \psi \int_D g_{L_\rho}(z, \zeta, D) d\nu(\zeta) = \int d\nu(\zeta) \int L_z^* \psi g_{L_\rho}(z, \zeta) dz$$

$$= \int_D \psi(\zeta) d\nu(\zeta) > 0,$$

so that $L_\rho \Pi(\cdot, \nu) = \nu$ in $\mathcal{D}'(\mathbb{T}_P^2)$.

Let $\nu'$ be the density of $\nu$. It is straightforward that for any $z_0 \in D \cup \partial D$

$$\sup_{|z-\zeta_0|<\epsilon} \int_{|\zeta-z_0|<\epsilon} |g_{L_\rho}(z, \zeta, D)| d\zeta \leq C \int_{|\zeta-z_0|<\epsilon} |\log|\zeta - z_0|| d\zeta$$

with a constant $C$ which does not depend on $\epsilon$ and $z_0$. Hence, $g_{L_\rho}(z_n, \zeta, D)\nu'(\zeta)$ is a Lebesgue sequence when $z_n \to z_0$. $\square$

Let $v$ be an $L_\rho$–s.f. Then since $v$ is upper semicontinuous, the solution $q(z)$ of the problem (7.5) with boundary data $v$ is defined for any regular domain $D$ for which $0 < \rho < \rho(D)$; this function $q = q(z) = q(z, f, D)$ is called *the least $L_\rho$–majorant of $v$ in $D$.*

**Proposition 8.9 (Riesz Theorem).** *Let $v$ and $q$ be as above, and $0 < \rho < \rho(D)$. Then*

$$v(z) = q(z) + \Pi(z, v),$$

*where $\nu = L_\rho v$ and $q$ is the least $L_\rho$–majorant of $v$ in $D$.*

*Proof.* If $v$ is sufficiently smooth, then $q = v - \Pi(z, \nu)$ is an $L_\rho$–function which agrees with $v$ on $\partial D$ quasi–everywhere. For an arbitrary $v$, we obtain this representation for a sequence $v_n \downarrow v$ of smooth $L_\rho$–s.f.'s, and take the limit in $\mathcal{D}'(\mathbb{T}_P^2)$. $\square$

It is clear that $q(z, v, D)$ is a majorant of $v$. The next assertion, which follows from Proposition 7.6, shows that it is the least majorant.

**Proposition 8.10.** *Let $D \subset \mathbb{T}_P^2$ be a domain and $0 < \rho < \rho(D)$. Let $f$ be upper semicontinuous on $\partial D$, and let $v$ be an $L_\rho$–s.f. in $\mathbb{T}_P^2$.*

*Then if $f \geq v$ on $\partial D$, it follows that $q = q(z, f, D) \geq v$ in $D$.*

**Theorem 8.11 (Sweeping).** *Let $\rho < \rho(D)$, $v$ an $L_\rho$–s.f. and $q(z, v, D)$ the least $L_\rho$–majorant of $v$. Then the function*

$$v(z, v, D) := \begin{cases} v(z) & (z \in \mathbb{T}_P^2 \backslash D) \\ q(z, v, D) & (z \in D). \end{cases}$$

*is an $L_\rho$–s.f.*

*Proof.* We need check this only in a neighborhood of $\partial D$ where it follows from the inequality

$$v(z) \leq q(z, v, D) \qquad (z \in D)$$

and Corollary 8.3(1). $\square$

**Example 8.12.** Consider the situation (0.2).We consider $v$ in the sector $G = \{z = re^{i\theta} : \alpha < \theta < \beta\}$ and associate to $G$ the subdomain $D = \{z \in \mathbb{T}_P^2 : \alpha \leq y \leq \beta\}$. Since $\rho(G) = \pi/(\beta - \alpha)$, we have that $\rho(D) = \pi/(\beta - \alpha)$ independent of $P$, and the condition $\rho < \rho(D)$ reduces to the classical requirement $(\beta - \alpha) < \pi/\rho$. If $h$ satisfies (0.1), then the smallest $\rho$–trigonometric majorant of $h$ on $(\alpha, \beta)$ is

$$H(\varphi) = \frac{h(\alpha)\sin\rho(\beta - \varphi) + h(\beta)\sin\rho(\varphi - \alpha)}{\sin\rho(\beta - \alpha)},$$

and Theorem 8.11 reduces to the so–called fundamental relation for $\rho$–t.c.functions [16, (1.69)]: if $\max\limits_{i,j} |\varphi_i - \varphi_j| < \pi/\rho$, $i, j = 1, 2, 3$, then

$$(8.13) \quad h(\varphi_1)\sin\rho(\varphi_2 - \varphi_3) + h(\varphi_2)\sin\rho(\varphi_3 - \varphi_1) + h(\varphi_3)\sin\rho(\varphi_1 - \varphi_2) \leq 0.$$

The procedure of constructing the least harmonic majorant $H(z, u, G)$ associated to a subharmonic function $u$ in $G \subset \mathbb{C}$ is called the 'sweeping of the masses' of $u$. Thus we may call our construction of $v(\cdot, v, D)$ the 'sweeping' of $L_\rho$–masses of $v$.

**8.14 Representation Theorems.** It is natural to describe new classes of functions in terms of independent parameters. For example, if $\rho$ is not an integer, a $\rho$–trigonometrically convex function $h$ has the representation

$$h(\varphi) = \frac{1}{2\rho\sin\pi\rho} \int_0^{2\pi} \cos^* \rho(\varphi - \psi - \pi)d\Delta(\psi)$$

where the function $\cos^* \rho\varphi$ is the $2\pi$–periodic extension of $\cos\rho\varphi$ from $(-\pi, \pi)$, and $\Delta(d\psi) = [h''(\psi) + \rho^2 h(\psi)]d\psi$ is a positive measure (see [16, Ch. 1, Theorem 24] and [6, Ch. 1], where the case $\rho \in \mathbb{N}$ is also considered). From our point of view, $d\Delta$ is the independent parameter for the class of $\rho$– t.c. functions, and its connection with the zero–distribution of functions of completely regular growth (Levin-Pfluger functions) is the theme of [16, Ch. 2].

We now consider the situation corresponding to the operator $L_\rho$ and on $\mathbb{T}_P^2$. Propositions 8.15, 8.17 and 8.19 generalize [16, Theorem 24]. The proof is new also for $\rho$–t.c.functions.

First, let $\rho \notin \mathbb{Z}$ and $E_\rho$ the fundamental solution of $L_\rho$ on $\mathbb{T}_P^2$, as in Proposition 1.1. For a measure $\nu$, consider the potential

$$\Pi_\rho(z, \nu) = \int_{\mathbb{T}_P^2} E_\rho(z - \zeta)d\nu(\zeta).$$

As in the proof of Proposition 8.8, $\Pi_\rho$ is an $L_\rho$–s.f. and $L_\rho\Pi_\rho = \nu$.

**Proposition 8.15.** *Let $\rho > 0$, $\rho \notin \mathbb{Z}$. Then every $L_\rho$–s.f. $v$ on $\mathbb{T}_P^2$ may be represented as*

$$v(z) = \Pi_\rho(z, \nu),$$

*where $\nu = L_\rho v$.*

**Lemma 8.16.** *Let $L_\rho q = 0$ on $\mathbb{T}_P^2$. Then $q \equiv 0$ when $\rho \notin \mathbb{Z}$, and $q(z) = \Re\{Ce^{ipy}\}$, $C \in \mathbb{C}$, when $\rho \in \mathbb{Z}$.*

*Proof.* Exactly as in Proposition 1.1, we see that the Fourier coefficients $\{q_{m,k}\}$ of $q$ must be chosen so that

$$q_{m,k}\left[-\left(\frac{2\pi m}{P}\right)^2 - k^2 - 2\rho i \frac{2\pi m}{P} + \rho^2\right] = 0, \ (m, k) \in \mathbb{Z}^2.$$

When $\rho \notin \mathbb{Z}$, this forces all $q_{m,k}$ to vanish, and when $\rho = p \in \mathbb{Z}$, the bracketed term vanishes when $m = 0$, $k = \pm p$. In this case, if $q_{0,\pm p} \neq 0$, we have $q = c_1 e^{-ipy} + c_2 e^{ipy}$, and since $q$ is real, the Lemma follows. $\square$

*Proof of Proposition 8.15.* We apply $L_\rho$ to $q \equiv v - \Pi_\rho$, and note that $q$ is an $L_\rho$–function on $\mathbb{T}_P^2$. Hence $q \equiv 0$ as follows from Lemma 8.16. $\square$

Consider now the case $\rho = p$. The following assertion generalizes the corresponding condition for the Phragmén-Lindelöf indicator [16, 1.81]. It describes a symmetry of mass distribution in the case of integral $\rho$.

**Proposition 8.17.** *Let $\rho = p \in \mathbb{Z}$, $p \geq 1$, $v$ be an $L_p$–s.f. on $\mathbb{T}_P^2$ and $\nu = L_p v$. Then*

$$(8.18) \qquad \int_{\mathbb{T}_P^2} e^{\pm ipy} d\nu(z) = 0.$$

*Proof.* The functions $e^{\pm ipy} \in \mathcal{D}(\mathbb{T}_P^2)$ and are solutions to the equation $L_p^* q = 0$ on $\mathbb{T}_P^2$. Thus

$$\int_{\mathbb{T}_P} e^{\pm ipy} d\nu(z) = \langle e^{\pm ip\cdot}, L_p v\rangle = \langle L_p^* e^{\pm ip\cdot}, v\rangle = 0. \quad \square$$

Let $E_p'(z)$ be arbitrary generalized fundamental solution from Proposition 1.10. Set

$$\Pi_p'(z, \nu) := \int_{\mathbb{T}_P^2} E_p'(z - \zeta) d\nu(\zeta).$$

The potential is defined uniquely because of Propositions 1.10 and 8.17.

The next assertion generalizes the representation of the Phragmén-Lindelöf indicator for functions of integral order [16, 1.82]

**Proposition 8.19.** *Let $p$ be an integer and $v$ be an $L_p$–s.f. on $\mathbb{T}_P^2$. Then*

$$(8.20) \qquad v(z) = \Pi_p'(z, \nu) + \Re(Ce^{ipy}),$$

*where $\nu = L_p v$, and $C$ is a complex scalar.*

*Proof.* Using Proposition 1.10 we have

$$(8.21) \qquad L_p \Pi_p'(z, \nu) = \nu - \int_{\mathbb{T}_P^2} \cos p \Im(z - \zeta) \, d\nu(\zeta).$$

Hence Proposition 8.17 gives $L_p \Pi_p'(z, \nu) = \nu$. Thus the function $q := v - \Pi_p'(z, \nu)$ satisfies $L_p q = 0$ on $\mathbb{T}_P^2$. Lemma 8.16 gives that $q = \Re(Ce^{ipy})$ $\square$

**9. $L_\rho$–subminorants.** An $L_\rho$–s.f. $v$ is called an $L_\rho$–subminorant ($L_\rho$–s.m.) of a real–valued function $m(z)$, $z \in D \subset \mathbb{T}_P^2$, if

$$(9.1) \qquad\qquad v(z) \leq m(z), \ z \in D.$$

An $L_\rho$–subminorant $v_0$ is called the *maximal $L_\rho$–subminorant* of $m$, if the conditions $\{w$ is an $L_\rho$–subminorant$\}$ and $\{w \geq v_0\}$ imply that $w = v_0$ in $D$.

**Theorem 9.2.** *If $m(z)$ is continuous and has an $L_\rho$–subminorant, then it has a unique maximal $L_\rho$–subminorant.*

*Proof.* This assertion follows by word–word repetition of the proof for the case of maximal subharmonic minorant [14], which we briefly sketch. The set of the subminorants is a partly ordered set, because the semicontinuous regularization of the supremum of any set of subminorants is also a subminorant. Hence, there exists a unique maximal element. $\square$

For some applications it is desirable to have maximal $L_\rho$–s.m. when $m(z)$ need not be continuous. At present, this is possible only in certain cases, even in the classical case of the Laplace operator. First, let $m(z)$ be upper semicontinuous, and let the sequence of continuous functions $m_n \downarrow m(z)$. The corresponding sequence of maximal $L_\rho$–s.m.'s decreases monotonically and thus converges to an $L_\rho$–s.f. $v_0$ which is the maximal $L_\rho$–s.m. for $m(z)$ as can readily be verified.

When $m(z)$ is not upper semicontinuous, we can, of course, consider its upper semicontinuous regularization

$$(9.3) \qquad\qquad m^*(z) := \lim_{\epsilon \to 0} \sup\{m(z) : |z - \zeta| < \epsilon\}$$

which is an upper semicontinuous function, and thus construct an $L_\rho$–s.m. for $m^*$. However, we cannot ensure that the maximal $L_\rho$–s.m. of $m^*$ does not exceed $m$ for all $z$.

We present some positive results.

**Theorem 9.4.** *Let $m = m_1 - m_2$ where $m_1, m_2$ are $L_\rho$–s.f.'s, and assume that $m$ has an $L_\rho$-subminorant. Then there exists a unique maximal $L_\rho$–s.m. $v_{\sup}^*$.*

*Proof.* Since $m$ has an $L_\rho$–s.m. $v$, we recall the notation (8.5) and observe that

$$\mathcal{M}_\epsilon(z, m) \equiv \mathcal{M}_\epsilon(z, m_1) - \mathcal{M}_\epsilon(z, m_2)$$

has the $L_\rho$–s.m. $\mathcal{M}_\epsilon(z, v)$ and so, by Theorem 9.2, has the unique maximal $L_\rho$–s.m. $v(z, \epsilon)$. Then

$$v_{\sup}(z) \equiv \limsup_{\epsilon \to 0} v(z, \epsilon) \leq m(z).$$

In this inequality, we refer to (9.3), and note that $v_{\sup}^*(z)$ is an $L_\rho$–s.f. that coincides with $v_{\sup}(z)$ everywhere except perhaps on a set of zero capacity. This follows by Cartan's theorem ([10], Ch.7) applied to the sequence of subharmonic functions $u(z, \epsilon) := v(\log |z|, \epsilon)|z|^\rho$. In general, $v_{\sup}^*$ can exceed $v_{\sup}(z)$. However, under our special hypotheses here, we claim that

$$(9.5) \qquad\qquad v_{\sup}^*(z) \leq m(z)$$

everywhere. Indeed, $m_2(z) + v^*_{\sup}(z) \le m_1(z)$ outside of a set of zero capacity, so

$$\mathcal{M}_\epsilon(z, m_2) + \mathcal{M}_\epsilon(z, v^*_{\sup}) \le \mathcal{M}_\epsilon(z, m_1).$$

The formula (8.5) now gives (9.5).

We show that $v^*_{\sup}$ is the maximal $L_\rho$–s.m. If not, there would exist an $L_\rho-$ s.m. $v_1$ exceeds $v^*_{\sup}$ on a set of positive measure (otherwise they coincide); thus we would have for some $z$ and $\epsilon$

$$v^*_{\sup}(z) < \mathcal{M}_\epsilon(z, v_1) \le v(z, \epsilon),$$

and this contradicts the definition of $v^*_{\sup}$.  $\square$

**Proposition 9.6.** *Let $v$ be the maximal $L_\rho$–s.m. of $m(z)$ such that $v(z) < m(z)$ on an open set $U$. Then $v$ is an $L_\rho$–function in $U$: $L_\rho v = 0$.*

This proof parallels that for a subharmonic function (see [14]), but we need some technical details.

*Proof of Proposition 9.6.* Note that in a disc $D_\delta := \{z \in \mathbb{T}^2_P : |z - z_0| < \delta\}$ such that $D_\delta \cap T D_\delta = \emptyset$ we have

$$q(z, v, D_\delta) = e^{-\rho x} \int\limits_{|\zeta - z_0| = \delta} P(z, \zeta, \delta) e^{\rho \xi} v(\zeta) |d\zeta|$$

where $P(\cdot, \cdot, \cdot)$ is the Poisson kernel, $|d\zeta|$ is the element of length. Since

$$\psi(\delta) := \max_{|z - \zeta| \le \delta} |e^{-\rho(z - \zeta)}| - 1 = o(\delta),$$

we obtain that

(9.7) $$q(z, v, D_\delta) \le (1 + o(\delta)) \max_{|\zeta - z_0| \le \delta} v(\zeta).$$

Suppose that $v(z_0) < m(z_0)$ and $v$ is not an $L_\rho$ -function in a neighborhood $U \ni z_0$. Set $m(z_0) - v(z_0) := d$. Choose a disc $D_\delta$ such that $\nu_v(D_\delta) > 0$ and so small that

(9.8) $$(1 + o(\delta)) \max_{\zeta \in D_\delta} v(\zeta) < v(z_0) + d/2,$$

where $o(\delta)$ is from (9.7). The inequality (9.8) is possible because of upper semicontinuity $v(z)$.

Let us replace $v$ in $D$ by its least $L_\rho$ -majorant, i.e. construct the sweeping $v(z, v, D)$ from Theorem 8.11. Then $v(z, v, D) < m(z)$ for $z \in D$ and hence for all $z \in \mathbb{T}^2_P$. But $v(z, v, D) > v(z)$ in $D_\delta$ because of the Riesz Theorem 8.9. Thus $v(z)$ is not the maximal minorant. This is a contradiction.  $\square$

**Proposition 9.9.** *If $m(z)$, $z \in \mathbb{T}^2_P$, is continuous, the maximal $L_\rho$–s.m. is continuous.*

Note from Proposition 9.6 that there is no problem at points where the maximal $L_\rho$–s.m. does not strictly exceed $m$.

We are going to use the following

**Theorem 9.10.** *Let $m(z), z \in \mathbb{C}$ be continuous and have a subharmonic minorant in $\mathbb{C}$. Then its maximal subharmonic minorant is continuous.*

This fact was not obvious for us and we could not find a proof. Thus we thank Prof. A. Eremenko for the following argument:

*Proof.* We prove continuity at $z = 1$. Let $m$ be the continuous function and $u$ its maximal subharmonic minorant. Since $u$ is already upper semicontinuous, we need only show that for every $\epsilon > 0$

$$(9.11) \qquad\qquad u(z) > u(1) - \epsilon$$

in some neighborhood of $z = 1$. Let $v$ be the sweeping of $u$ in a neighborhood $U$ of $1$ (in a small disc). Then it is easy to see that $u(z) \leq v(z) < m(1)+\epsilon/4 < m(z)+\epsilon/2$ in $U$. Hence $v-\epsilon$ is a subharmonic minorant of $m$, and so $u > v-\epsilon$ everywhere. Since $v$ is continuous in the disk, we can find a neighborhood of $z$ in which $v(z) > v(1)-\epsilon/2$. Thus (9.11) holds in this neighborhood.

**Proposition 9.12.** *Let $m$ be continuous in $\mathbb{C}$ and satisfy the condition (8.2). Then its maximal subharmonic minorant also satisfies (8.2).*

*Proof.* Let $u(z)$ be a subharmonic minorant of $m(z)$. Then

$$u_1(z) := [\sup_{n \in \mathbb{Z}} u(e^{nP} z)e^{-\rho nP}]^*,$$

where $[\cdot]^*$ is defined by (9.3), is a subharmonic minorant of $m(z)$, $u_1(z) \geq u(z)$, $z \in \mathbb{C}$, and $u_1$ satisfies (8.2).     □

*Proof of Proposition 9.9.* Set $m_1(\lambda) := m(\log \lambda)|\lambda|^\rho$. It is continuous in $\mathbb{C}$ and satisfies the assumption of Proposition 9.12. Thus its maximal subharmonic minorant $v_1(\lambda)$ is continuous and satisfies (8.2), so by Proposition 8.1, $v(z) := v_1(e^z)e^{-\rho x}$ is the $L_\rho$–s.m. of $m$. In particular $v$ is continuous.     □

If $m(z)$ is not continuous or even upper semicontinuous, its maximal $L_\rho$–s.m. may still be continuous.

**Theorem 9.13.** *Let $m = m_1 - m_2$ where $m_1, m_2$ are $L_\rho$–s.f.'s. Then the maximal $L_\rho$–s.m. is continuous if $m_1$ is continuous.*

*Proof.* Let $v$ be the maximal $L_\rho$–s.m. of $m$, and let $\nu_m = \nu_{m_1} - \nu_{m_2}$, where $\nu_{m_i} = L_\rho m_i$, $i = 1, 2$. Set $g(z) = e^{\rho x}[m_1 - m_2 - v](z)$ and let $E = \{z : g(z) = 0\}$. On $\mathbb{C} \setminus E$ we have $g(z) > 0$.

Now we use the following assertion (Grishin Lemma)

**Theorem AFG** [7]**.** *Let $g$ be a nonnegative $\delta$ -subharmonic function and $\nu_g$ be its charge. Then the restriction $\nu_g|_E$ to the set $E := \{z : g(z) = 0\}$ is a measure.*

Thus

$$\nu_v \leq \nu_{m_1} - \nu_{m_2} \leq \nu_{m_1}.$$

on E. By Proposition 9.6 $\nu_v = 0$ outside $E$. Hence this also holds in $\mathbb{C}$.

It follows from Jensen's theorem that a subharmonic function $u$ is continuous at a point $z_0$ if and only if

$$\int\limits_0^\epsilon \frac{\mu_u\{z : |z - z_0| < t\}}{t} dt = o(1), \epsilon \to 0.$$

By hypothesis $m_1$ and, hence, $u := e^{\rho x} m_1$ are continuous. Thus $e^{\rho x} v$ is continuous, and hence so is $v$. $\square$

**9.14. A new set characteristic.** Let $D \subset \mathbb{T}_P^2$ be a domain, $\rho(D)$ be its order, and set

$$\lambda(D) = \frac{1}{\rho(D)}.$$

This characteristic is a "natural" monotonic functional and is zero on any domain which is not connected on spirals. We extend $\lambda$ to arbitrary sets in the standard way. If $D \subset \mathbb{T}_P^2$ is open, define $\lambda(D) = \max\limits_i \lambda(D_i)$, where $\{D_i\}$ are the connected components of $D$. For a closed set $K \subset \mathbb{T}_P^2$ we define $\lambda$ as

$$\lambda(K) = \inf_{D \supset K} \lambda(D),$$

where $D$ is open. Finally, for any set $E \subset \mathbb{T}_P^2$ let

$$\overline{\lambda}(E) = \inf_{D \supset E} \lambda(D); \ \underline{\lambda}(E) = \sup_{K \subset E} \lambda(K),$$

where $\{D\}$ are open and $\{K\}$ are closed.

It is important to know if a given function $m(z)$ has an $L_\rho$–s.m. and then describe its maximal $L_\rho$–s.m. (cf.[3, 4]):

**Theorem 9.15.** *Let $m(z)$ be a function on $\mathbb{T}_P^2$, and let*

$$E^+(m) = \{z \in \mathbb{T}_P^2, m(z) > 0\}.$$

*If $m$ has a non-zero $L_\rho$–subminorant, then*

$$\overline{\lambda}(E^+(m)) \geq \frac{1}{\rho}.$$

**Theorem 9.16.** *Let $m(z) \geq 0$ be a continuous function and*

$$\lambda(E^+(m)) > \frac{1}{\rho}.$$

*Then $m(z)$ has a non-zero $L_\rho$–subminorant.*

Theorem 9.15 follows directly from

**Theorem 9.17.** *Let $v(z)$, $z \in \mathbb{T}_P^2$ be an $L_\rho$–s.f. and $E^+(v)$ be defined as in the statement of Theorem 9.15. Then $\overline{\lambda}(E^+(v)) \geq 1/\rho$ or $v \equiv 0$ in $\mathbb{T}_P^2$.*

*Proof.* Suppose the theorem false, and choose an open set $D \supset E^+(v)$ such that $\lambda(D) < 1/\rho$. Hence for each component $D_i$ of $D$, $\rho(D_i) > \rho$. Since the function $q \equiv 0$ is the unique solution of the boundary problem (0.15) in each $D_i$ we obtain from Theorem 8.10 that $v(z) \leq 0$, $z \in D_i$. Thus $v(z) = 0, \zeta \in \mathbb{T}_P^2$ by Proposition 8.6(2). $\square$

For the proof of Theorem 9.16 we need

**Lemma 9.18.** *For every domain $D$ with $\rho(D) < \rho$ there exists a domain $D(\rho) \subset\subset$ $D$ such that $\rho(D(\rho)) = \rho$.*

This follows from Theorem 0.17 and Proposition 6.6.

*Proof of Theorem 9.16.* Since $E^+(m)$ is open, the condition $\lambda(E^+(m)) > \rho^{-1}$ implies there is a connected component $D \subset E^+(m)$ with $\rho(D) < \rho$. Let $D_1 := D(\rho)$ be from Lemma 9.18 and $v_1$ be a solution to (0.15). Set

$$v(z) = \begin{cases} Cv_1(z), & z \in D_1 \\ 0, & z \notin \mathbb{T}^2_P \backslash D_1, \end{cases}$$

with $C < \min\limits_{z \in D_1} m(z)/v_1(z)$. Then $Cv_1(z) < m(z)$ for $z \in D_1$. We thus obtain an $L_\rho$–s.m. of $m(z)$.  □

Here is another necessary condition.

**Proposition 9.19.** *Let $m(z)$, $z \in \mathbb{T}^2_P$ have an $L_\rho$–s.m. ($\rho > 0$). Then*

$$(9.20) \qquad \int_0^{2\pi} m(x + iy)dy \geq 0, \ \forall x.$$

*Proof.* Let $v(z)$ be an $L_\rho$–s.m., and associate to $v$ the subharmonic function $V(z) = v(\log z)|z|^\rho$. Since $V(0) = 0$ we have that $\int\limits_0^{2\pi} V(re^{i\phi})d\phi \geq v(0) = 0, \ \forall r > 0$. Therefore (9.20) holds for $v$ and hence for $m$.

**9.21. Minimality.** An $L_\rho$–s.f. $v$ is called *minimal* if $v - \varepsilon$ does not have an $L_\rho$–s.m. for any $\varepsilon > 0$, (see also [3, 4]). We shall have many examples of minimal $L_\rho$–s.f. in $\mathbb{T}^2_P$ once we establish the following theorem.

Denote by $\mathcal{H}_\rho(v)$ the maximal open set in which $L_\rho v = 0$; i.e., $v$ is an $L_\rho$–function in $\mathcal{H}_\rho(v)$.

**Theorem 9.22=0.20.** *If there exists a connected component $D \subset \mathcal{H}_\rho(v)$ such that $\rho(D) < \rho$, then $v$ is a minimal $L_\rho$–s.f.*

For example, $v \equiv 0$ is a minimal $L_\rho$–s.f. because there cannot be a negative $L_\rho$–s.f. in $\mathbb{T}^2_P$ because of Proposition 8.6(2).

*Proof.* We first note that if $\rho(D) < \rho$, an $L_\rho$–s.f. $v$ cannot be negative in all of $D$. Indeed, let $q > 0$ solve (0.15) in a domain $D_1 \subset\subset D$ such that $\rho(D_1) = \rho$. Choose $C = \min\limits_{z \in D_1} v(z)/(-q(z))$ so that $v(z_0) + Cq(z_0) = 0$ for some $z_0 \in D_1$ and $v(z) + Cq(z) \leq 0$ in $D_1$, but this contradicts the maximum principle Theorem 8.6(2).

Now suppose the Theorem is false, and let $v'$ be an $L_\rho$–s.m. of $v - \varepsilon$. Then the function $v' - v$ is an $L_\rho$–s.f. which is negative in $D$. Theorem 9.22 follows.  □

It is possible to produce sufficient conditions for nonminimality. For example,

**Proposition 9.23.** *The function $v$ is nonminimal if $v(z) \geq c$ or $L_\rho v - c > 0$ for some positive $c$ for all $z \in \mathbb{T}^2_P$.*

This Proposition follows since $v \equiv c$ is an $L_\rho$–s.f. But a complete characterization of minimal $L_\rho$–subfunctions remains open (see [8, Problem 16.9]).

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